

Lipschitz Estimates in Almost-Periodic Homogenization

Scott N. Armstrong and Zhongwei Shen*

Abstract

We establish uniform Lipschitz estimates for second-order elliptic systems in divergence form with rapidly oscillating, almost-periodic coefficients. We give interior estimates as well as estimates up to the boundary in bounded $C^{1,\alpha}$ domains with either Dirichlet or Neumann data. The main results extend those in the periodic setting due to Avellaneda and Lin [2, 5] for interior and Dirichlet boundary estimates and later Kenig, Lin, and Shen [14] for the Neumann boundary conditions. In contrast to these papers, our arguments are constructive (and thus the constants are in principle computable) and the results for the Neumann conditions are new even in the periodic setting, since we can treat non-symmetric coefficients. We also obtain uniform $W^{1,p}$ estimates.

Keywords: homogenization; almost-periodic coefficients; Lipschitz estimates.

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1 Introduction

The primary purpose of this paper is to establish uniform Lipschitz estimates for a family of elliptic operators with rapidly oscillating, almost-periodic coefficients, arising in the theory of homogenization. More precisely, we consider the linear elliptic operator

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_j} \right\}, \quad \varepsilon > 0 \quad (1.1)$$

(the summation convention is used throughout). Let $A(y) = (a_{ij}^{\alpha\beta}(y))$ be real and bounded in \mathbb{R}^d , where $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$. Throughout the paper we will assume that

$$\mu|\xi|^2 \leq a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \leq \mu^{-1}|\xi|^2 \quad \text{for any } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{m \times d}, \quad (1.2)$$

where $\mu > 0$, and

$$\lim_{R \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq R}} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)} = 0. \quad (1.3)$$

Notice that if A is bounded and continuous in \mathbb{R}^d , then A satisfies (1.3) if and only if A is *uniformly almost-periodic* in \mathbb{R}^d , i.e., each entry of A is the uniform limit of a sequence of trigonometric polynomials. We define the following modulus, which quantifies the almost periodic assumption:

$$\rho(R) := \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq R}} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}. \quad (1.4)$$

Given a bounded $C^{1,\alpha}$ domain $\Omega \subset \mathbb{R}^d$, we are interested in estimating the quantity $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)}$, uniformly in $\varepsilon > 0$, for weak solutions u^ε of Dirichlet problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega, \quad (1.5)$$

as well as those of the Neumann problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega. \quad (1.6)$$

In (1.6) we have used $\partial u_\varepsilon / \partial \nu_\varepsilon$ to denote the conormal derivative $n(x)A(x/\varepsilon)\nabla u_\varepsilon(x)$ on $\partial\Omega$, where $n(x)$ is the outward unit normal to $\partial\Omega$.

To ensure that we have Lipschitz estimates at small scales, we assume that A is uniformly Hölder continuous, i.e., there exist $\tau > 0$ and $\lambda \in (0, 1]$ such that

$$|A(x) - A(y)| \leq \tau |x - y|^\lambda \quad \text{for any } x, y \in \mathbb{R}^d. \quad (1.7)$$

The following are the main results of the paper.

Theorem 1.1. *Suppose that $A(y)$ satisfies uniform ellipticity (1.2) and Hölder continuity conditions (1.7). Suppose also that there exist $N > 5/2$ and $C_0 > 0$ such that*

$$\rho(R) \leq C_0 [\log R]^{-N} \quad \text{for any } R \geq 2. \quad (1.8)$$

Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d for some $\alpha > 0$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of Dirichlet problem (1.5). Then

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq C \{ \|F\|_{L^p(\Omega)} + \|f\|_{C^{1,\beta}(\partial\Omega)} \}, \quad (1.9)$$

where $p > d$, $\beta \in (0, \alpha)$, and C depends only on p , β , A , and Ω .

Theorem 1.2. *Suppose that $A(y)$ satisfies (1.2) and (1.7). Also assume that the decay condition (1.8) holds for some $N > 3$ and $C_0 > 0$. Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d for some $\alpha > 0$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of the Neumann problem (1.6). Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq C \{ \|F\|_{L^p(\Omega)} + \|g\|_{C^\beta(\partial\Omega)} \}, \quad (1.10)$$

where $p > d$, $\beta \in (0, \alpha)$, and C depends only on p , β , A , and Ω .

Note that if $A(y)$ is periodic, then $\rho(R) = 0$ for R sufficiently large and thus satisfies the assumption (1.8) for any $N > 1$. In this case the Lipschitz estimate (1.9) for the Dirichlet problem (1.5) in $C^{1,\alpha}$ domains was established by Avellaneda and Lin [2] under the conditions (1.2) and (1.7). This classical result was recently extended by Kenig, Lin, and Shen in [14], where estimate (1.10) was established for solutions of the Neumann problem (1.6) in the periodic setting, under an additional symmetry condition $A^*(y) = A(y)$, i.e., $a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y)$ for any $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$. Our Theorems 1.1 and 1.2 further extend the main results in [2] and [14] to the almost-periodic setting. We point out that Theorem 1.2 is new even in the periodic setting, as the symmetry condition $A^* = A$ is not required. We also remark that the Lipschitz estimates in Theorems 1.1 and 1.2 are sharp in the sense that there is no uniform modulus of continuity for the gradient of solutions unless $\text{div}(A) = 0$. As for the $C^{1,\alpha}$ assumption on the domain Ω , we note that Lipschitz estimates may fail on a C^1 domain even for harmonic functions.

The proof of uniform estimates in both [2] and [14] is based on a compactness argument that originated from the study of regularity theory in the calculus of variation and minimal surfaces. The argument, which was introduced in [2] to the study of homogenization, extends readily to the almost-periodic setting in the case of uniform Hölder estimates. In fact it was proved in [21] that if u_ε is a weak solution of the Dirichlet problem:

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F + \text{div}(h) \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega, \quad (1.11)$$

where Ω is a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d , then

$$\begin{aligned} \|u_\varepsilon\|_{C^\beta(\overline{\Omega})} \leq C \Bigg\{ & \|f\|_{C^\beta(\partial\Omega)} + \sup_{\substack{x \in \Omega \\ 0 < r < r_0}} r^{2-\beta} \int_{B(x,r) \cap \Omega} |F| \\ & + \sup_{\substack{x \in \Omega \\ 0 < r < r_0}} r^{1-\beta} \left(\int_{B(x,r) \cap \Omega} |h|^2 \right)^{1/2} \Bigg\} \end{aligned} \quad (1.12)$$

for any $\beta \in (0, 1)$, where $r_0 = \text{diam}(\Omega)$ and C depends only on β , A , and Ω . However, for Lipschitz estimates, the approach in [2, 14] relies on the Lipschitz estimates for interior and boundary correctors in a crucial way. It is not clear how to extend this to the almost-periodic setting, as any estimate of correctors in a non-periodic setting is far from trivial, even in the interior case.

Our proof of Theorem 1.1 and 1.2 will be based on a rather general scheme for proving Lipschitz estimates at large scale in homogenization. The scheme, which was motivated by the compactness argument in [2], was recently formulated and used by the first author and C. Smart in [1] for convex integral functionals with random coefficients. The idea, rather than arguing by contradiction (by compactness), is to apply a $C^{1,\alpha}$ Campanato iteration directly. For this we need to show that the “flatness” of a solution u (how well it is approximated by an affine function) improves on smaller scales, e.g., for some $\theta \in (0, 1/4)$,

$$\begin{aligned} & \frac{1}{r\theta} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_{\theta r}} |u(x) - Mx - q|^2 dx \right)^{1/2} \\ & \leq \frac{1}{2} \left(\frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_r} |u(x) - Mx - q|^2 dx \right)^{1/2} \right). \end{aligned} \quad (1.13)$$

Since solutions of the *homogenized* equation satisfy such an estimate (on all scales), we indeed have (1.13) up to the error arising in homogenization. For large balls, we may expect this error to be much smaller than the improvement in the flatness. Therefore, if we can control the error in homogenization effectively, we may hope to iterate the improvement of flatness estimate down to microscopic scales. Indeed, as we show in Theorem 3.2 (which is a slight modification of [1, Lemma 5.1]), this scheme yields a uniform Lipschitz estimate down to the microscopic scale, provided that the rate of homogenization is sufficiently fast: an algebraic (or even Dini-type) convergence rate suffices.

Such error estimates were recently proved by the second author [21] for solutions u_ε of Dirichlet problem (1.5). In particular, it was shown that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \omega(\varepsilon) \|u_0\|_{W^{2,p}(\Omega)}, \quad (1.14)$$

where $p > d$ and $\omega(\varepsilon)$ is a modulus on $(0, 1]$, with $\omega(0+) = 0$, which can be given explicitly using the modulus $\rho(R)$ in (1.4) (see Section 2). The decay conditions on $\rho(R)$

in Theorems 1.1 and 1.2 are used precisely to ensure that we have Dini-type rates for homogenization. We mention that condition (1.8) holds, for example, if $A(y)$ is quasi-periodic in \mathbb{R}^d with frequencies satisfying the so-called Kozlov (C) condition [15]. In fact it was proved in [21] that the Kozlov (C) condition implies that $\rho(R) \leq C_0(R+1)^{-\lambda}$ for some $\lambda > 0$.

We do not know if the decay assumptions on $\rho(R)$ in Theorems 1.1 and 1.2 can be weakened substantially or whether uniform Lipschitz estimates hold for general uniformly almost periodic coefficients. However, we remark that the scheme for proving Lipschitz estimates formalized in Theorem 3.2 is a quite general tool that can be useful in other circumstances. It applies, for example, to the Poisson equation

$$-\Delta u = \operatorname{div} f \tag{1.15}$$

and yields a Lipschitz estimate on u precisely in the case that f is C^α (or Dini continuous). Likewise, a straightforward modification of Theorem 3.2 yields a statement that implies the classical Schauder estimates. Of course, if f is merely continuous, then it is well-known that solutions of (1.15) may fail to be Lipschitz continuous. This suggests that, in analogy, the decay conditions on $\rho(R)$ are natural and perhaps even necessary.

The general scheme mentioned above makes minimal use of the structure of the equation and in particular does not involve correctors in a direct manner (though indirectly via approximation requirements). As a result, it can be adapted surprisingly well for proving Lipschitz estimates up to the boundary with either Dirichlet or Neumann boundary conditions. The key step then is to establish suitable error estimates of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$, not necessarily sharp, for local weak solutions with Dirichlet or Neumann condition. This will be achieved by considering the function

$$w_\varepsilon = u_\varepsilon(x) - v_0(x) - \varepsilon \chi_T(x/\varepsilon) \nabla v_0(x),$$

where $T = \varepsilon^{-1}$, $\mathcal{L}_0(v_0) = 0$, and $\chi_T(y)$ denotes the approximate correctors for \mathcal{L}_ε . The proof relies on the pointwise estimates of χ_T obtained in [21]. In the case of Neumann conditions our argument also requires uniform Lipschitz estimates of χ_T , which follow from uniform interior Lipschitz estimates. However, as we indicated earlier, our approach does not use boundary correctors.

Let $G_\varepsilon(x, y)$ denote the matrix of Green functions for \mathcal{L}_ε in Ω , with pole at y . It follows from the proof of Theorem 1.1 that for any $x, y \in \Omega$ and $x \neq y$,

$$|\nabla_x G_\varepsilon(x, y)| + |\nabla_y G_\varepsilon(x, y)| \leq C |x - y|^{1-d} \tag{1.16}$$

and

$$|\nabla_x \nabla_y G_\varepsilon(x, y)| \leq C |x - y|^{-d}, \tag{1.17}$$

where C depends only on A and Ω . This, in particular, implies that the Poisson kernel $P_\varepsilon(x, y)$ for \mathcal{L}_ε in Ω satisfies

$$|P_\varepsilon(x, y)| \leq \frac{C \operatorname{dist}(x, \partial\Omega)}{|x - y|^d} \tag{1.18}$$

for any $x \in \Omega$ and $y \in \partial\Omega$. As in the periodic setting [3, 2], estimate (1.18) yields the following.

Theorem 1.3. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.1. Let $1 < p < \infty$. Let u_ε be the solution of the L^p Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega \quad (1.19)$$

with $(u_\varepsilon)^ \in L^p(\partial\Omega)$, where $f \in L^p(\partial\Omega; \mathbb{R}^m)$ and $(u_\varepsilon)^*$ denotes the non-tangential maximal function of u_ε . Then*

$$\|(u_\varepsilon)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}, \quad (1.20)$$

where C_p depends only on p , A , and Ω . Furthermore, if $f \in L^\infty(\partial\Omega)$, then

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\partial\Omega)}, \quad (1.21)$$

where C depends only on A and Ω .

In this paper we also study the uniform $W^{1,p}$ estimates for \mathcal{L}_ε . Related results in the periodic setting may be found in [2, 5, 20, 10, 14, 9]. We emphasize that the Hölder condition (1.7) is not assumed in the following two theorems.

Theorem 1.4. *Suppose that $A(y)$ is uniformly almost-periodic in \mathbb{R}^d and satisfies (1.2). Also assume that $A(y)$ satisfies the condition (1.8) for some $N > (3/2)$. Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d for some $\alpha > 0$ and $1 < p < \infty$. Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ be a weak solution of the Dirichlet problem (1.11), where $h = (h_i^\alpha) \in L^p(\Omega; \mathbb{R}^{m \times d})$, $F \in L^p(\Omega; \mathbb{R}^m)$ and $f \in B^{p,1-\frac{1}{p}}(\partial\Omega; \mathbb{R}^m)$. Then*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C_p \left\{ \|h\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|f\|_{B^{p,1-\frac{1}{p}}(\partial\Omega)} \right\}, \quad (1.22)$$

where C_p depends only on p , A , and Ω .

Theorem 1.5. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.4. Let $1 < p < \infty$. Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h) + F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g - n \cdot h \quad \text{on } \partial\Omega. \quad (1.23)$$

Then

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C_p \left\{ \|h\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{B^{-\frac{1}{p},p}(\partial\Omega)} \right\}, \quad (1.24)$$

where C_p depends only on p , A , and Ω .

We conclude this section with some notation and comments on bounding constants C . We will use $\int_E f = \frac{1}{|E|} \int_E f$ to denote the L^1 average of f over a set E . For a ball $B = B(x, r)$ we use αB to denote $B(x, \alpha r)$. We will use C to denote constants that may depend on d , m , $A(y)$, Ω , and other relevant parameters, but never on ε . It is important to note that since our assumptions on A are invariant under translation and rotation, the constants C will be invariant under any translation and rotation of Ω . This allows us to use freely translation and rotation to simplify the argument. As for rescaling, we observe that if $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ and $v(x) = u_\varepsilon(rx)$, then $\mathcal{L}_{\varepsilon/r}(v) = G$, where $G(x) = r^2 F(rx)$.

2 Homogenization and convergence rates

Let $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$. Throughout this section we assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$ is uniformly almost-periodic in \mathbb{R}^d and satisfies the ellipticity condition (1.2). The Hölder continuity (1.7) and decay condition (1.8) will not be used here.

2.1 The homogenized operator and qualitative homogenization

To define the homogenized operator \mathcal{L}_0 , we first introduce the space $B^2(\mathbb{R}^d)$, the L^2 space of almost-periodic functions in the sense of Bezikovich.

A function f in $L^2_{\text{loc}}(\mathbb{R}^d)$ is said to belong to $B^2(\mathbb{R}^d)$ if f is a limit of a sequence of trigonometric polynomials in \mathbb{R}^d with respect to the semi-norm

$$\|f\|_{B^2} = \limsup_{R \rightarrow \infty} \left\{ \int_{B(0,R)} |f|^2 \right\}^{1/2}.$$

For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, a number $\langle f \rangle$ is called the mean value of f if

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} f(x/\varepsilon) \varphi(x) dx = \langle f \rangle \int_{\mathbb{R}^d} \varphi$$

for any $\varphi \in C_0^\infty(\mathbb{R}^d)$. It can be shown that if $f, g \in B^2(\mathbb{R}^d)$, then fg has the mean value. Under the equivalent relation that $f \sim g$ if $\|f - g\|_{B^2} = 0$, the vector space $B^2(\mathbb{R}^d)/\sim$ becomes a Hilbert space with the inner product defined by $(f, g) = \langle fg \rangle$.

Let V_{pot}^2 (reps. V_{sol}^2) denote the closure in $B^2(\mathbb{R}^d; \mathbb{R}^{m \times d})$ of potential (reps. solenoidal) trigonometric polynomials with mean value zero. Then

$$B^2(\mathbb{R}^d; \mathbb{R}^{m \times d}) = V_{\text{pot}}^2 \oplus V_{\text{sol}}^2 \oplus \mathbb{R}^{m \times d}.$$

By the Lax-Milgram Theorem and the ellipticity condition (1.2), for any $1 \leq j \leq d$ and $1 \leq \beta \leq m$, there exists a unique $\psi_j^\beta = (\psi_{ij}^{\alpha\beta}) \in V_{\text{pot}}^2$ such that

$$\langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \phi_i^\alpha \rangle = -\langle a_{ij}^{\alpha\beta} \phi_i^\alpha \rangle \quad \text{for any } \phi = (\phi_i^\alpha) \in V_{\text{pot}}^2. \quad (2.1)$$

Let $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$, where

$$\widehat{a}_{ij}^{\alpha\beta} = \langle a_{ij}^{\alpha\beta} \rangle + \langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \rangle. \quad (2.2)$$

The homogenized operator for \mathcal{L}_ε is given by $\mathcal{L}_0 = -\operatorname{div}(\widehat{A}\nabla)$. We refer the reader to [12] for details (also see earlier work in [15, 16, 17]).

The proof of the following theorem may be found in [12].

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . For $F \in H^{-1}(\Omega; \mathbb{R}^m)$ and $\varepsilon > 0$, let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω . Suppose that for some subsequence $\{u_{\varepsilon'}\}$, $u_{\varepsilon'} \rightarrow u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and $A(x/\varepsilon')\nabla u_{\varepsilon'} \rightarrow G$ weakly in $L^2(\Omega; \mathbb{R}^{m \times d})$. Then $G = \widehat{A}\nabla u_0$ in Ω .*

The homogenization of Dirichlet problem (1.5) and the Neumann problem (1.6) follows readily from Theorem 2.1. For $\varepsilon \geq 0$, $F \in H^{-1}(\Omega; \mathbb{R}^m)$ and $f \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$, let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the unique weak solution of (1.5). Then $u_\varepsilon \rightarrow u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and strongly in $L^2(\Omega; \mathbb{R}^m)$, as $\varepsilon \rightarrow 0$. Similarly, if $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$, the solution of the Neumann problem (1.6) with $F \in H^{-1}(\Omega; \mathbb{R}^m)$ and $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^m)$ converges weakly in $H^1(\Omega; \mathbb{R}^m)$ to the solution of the Neumann problem: $\mathcal{L}_0(u_0) = F$ in Ω and $\partial u_0 / \partial \nu_0 = g$ on $\partial\Omega$, where $\partial u_0 / \partial \nu_0 = n \hat{A} \nabla u_0$.

2.2 Quantitative estimates for the approximate correctors

To study the convergence rates of u_ε to u_0 , we need to introduce the approximate correctors $\chi_T = (\chi_{T,j}^\beta)$. Let $P_j^\beta(y) = y_j e^\beta$, where $1 \leq j \leq d$, $1 \leq \beta \leq m$, and $e^\beta = (0, \dots, 1, \dots, 0)$ with 1 in the β^{th} position. For each $T > 0$, the function $u = \chi_{T,j}^\beta$ is defined as the weak solution of

$$-\operatorname{div}(A(y)\nabla u) + T^{-2}u = \operatorname{div}(A(y)\nabla P_j^\beta) \quad \text{in } \mathbb{R}^d, \quad (2.3)$$

with the property

$$\sup_{x \in \mathbb{R}^d} \|u\|_{H^1(B(x,1))} < \infty.$$

It is not hard to show that

$$\sup_{x \in \mathbb{R}^d} \int_{B(x,T)} (|\nabla \chi_T|^2 + T^{-2}|\chi_T|^2) \leq C, \quad (2.4)$$

where C depends only on d , m , and μ (the almost-periodicity of A is not needed).

For $\sigma \in (0, 1]$ and $T \geq 1$, define

$$\Theta_\sigma(T) = \inf_{0 < R \leq T} \left\{ \rho(R) + \left(\frac{R}{T} \right)^\sigma \right\}. \quad (2.5)$$

The following theorem was proved in [21].

Theorem 2.2. *Suppose that A is uniformly almost-periodic in \mathbb{R}^d and satisfies (1.2). Let $\sigma \in (0, 1)$ and $T \geq 1$. Then, for any $x, y \in \mathbb{R}^d$,*

$$|\chi_T(x) - \chi_T(y)| \leq C_\sigma T^{1-\sigma} |x - y|^\sigma, \quad (2.6)$$

and

$$T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \leq C_\sigma \Theta_\sigma(T), \quad (2.7)$$

where C_σ depends only on σ and A .

The rest of this section is devoted to the study of error estimates of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$. The material is divided into two subsections. The first subsection treats Dirichlet boundary condition, while the second handles the Neumann boundary condition.

2.3 Convergence rates: Dirichlet boundary condition

We begin by using Theorem 2.2 to extend a result in [21].

Lemma 2.3. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the weak solution of (1.5). Let*

$$w_\varepsilon = u_\varepsilon - v_0 - \varepsilon \chi_T(x/\varepsilon) \nabla v_0 - v_\varepsilon, \quad (2.8)$$

where $T = \varepsilon^{-1}$, $v_0 \in W^{2,2}(\Omega; \mathbb{R}^m)$, $\mathcal{L}_0(v_0) = F$ in Ω , and $v_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ is the weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = 0 & \text{in } \Omega, \\ v_\varepsilon = u_\varepsilon - v_0 - \varepsilon \chi_T(x/\varepsilon) \nabla v_0 & \text{on } \partial\Omega. \end{cases}$$

Then, for any $\sigma \in (0, 1)$,

$$\|w_\varepsilon\|_{H^1(\Omega)} \leq C_\sigma \{ \Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle \} \{ \|\nabla^2 v_0\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)} \}, \quad (2.9)$$

where $\psi = (\psi_{ij}^{\alpha\beta})$ is defined by (2.1) and C_σ depends only on σ , A , and Ω .

Proof. The proof is similar to that of Theorem 7.3 in [21], where v_0 is taken to be u_0 . A direct computation shows that

$$\mathcal{L}_\varepsilon(w_\varepsilon) = -\operatorname{div}(B_T(x/\varepsilon) \nabla v_0) + \varepsilon \operatorname{div}\{A(x/\varepsilon) \chi_T(x/\varepsilon) \nabla^2 v_0\},$$

where $B_T(y) = (b_{T,ij}^{\alpha\beta})$ is given by

$$b_{T,ij}^{\alpha\beta}(y) = \widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(y) - a_{ik}^{\alpha\gamma}(y) \frac{\partial}{\partial y_k} \{ \chi_{T,j}^\beta(y) \}. \quad (2.10)$$

Since $w_\varepsilon = 0$ on $\partial\Omega$, it follows that

$$c \int_\Omega |\nabla w_\varepsilon|^2 \leq \left| \int_\Omega \operatorname{div}(B_T(x/\varepsilon) \nabla v_0) \cdot w_\varepsilon \, dx \right| + \int_\Omega |\varepsilon \chi_T(x/\varepsilon)| |\nabla^2 v_0| |\nabla w_\varepsilon| \, dx. \quad (2.11)$$

Thus, it suffices to show that the right hand side of (2.11) is bounded by

$$C_\sigma \{ \Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle \} \{ \|\nabla^2 v_0\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)} \} \|w_\varepsilon\|_{H^1(\Omega)}$$

for any $\sigma \in (0, 1)$. By (2.7) and Cauchy inequality, the second integral in the right hand side of (2.11) is bounded by

$$C_\sigma \Theta_\sigma(T) \|\nabla^2 v_0\|_{L^2(\Omega)} \|\nabla w_\varepsilon\|_{L^2(\Omega)}.$$

The estimate of the first integral is much more delicate and is done by the same argument as in the proof of Theorem 7.3 in [21]. The key idea is to solve the equation

$$-\Delta H + T^{-2}H = B_T - \langle B_T \rangle \quad \text{in } \mathbb{R}^d, \quad (2.12)$$

and show that there exists a solution $H = H_T = \left(h_{ij}^{\alpha\beta}\right) \in W_{loc}^{2,2}(\mathbb{R}^d)$ satisfying

$$\begin{cases} T^{-2}\|H\|_{\infty} \leq C \Theta_1(T), \\ T^{-1}\|\nabla H\|_{\infty} \leq C_{\sigma} \Theta_{\sigma}(T), \end{cases} \quad (2.13)$$

and

$$\left\| \nabla \frac{\partial h_{ij}^{\alpha\beta}}{\partial x_i} \right\|_{\infty} \leq C_{\sigma} \Theta_{\sigma}(T) \quad (2.14)$$

for any $\sigma \in (0, 1)$ (the index i in (2.14) is summed from 1 to d). We omit the details. A similar approach will be used in the proof of Lemma 2.7. \square

Lemma 2.4. *Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d for some $\alpha > 0$. Let $u_{\varepsilon} \in H^1(\Omega; \mathbb{R}^m)$ ($\varepsilon \geq 0$) be the weak solution of (1.5). Then, for any $\sigma, \delta \in (0, 1)$,*

$$\begin{aligned} \|u_{\varepsilon} - u_0\|_{L^2(\Omega)} &\leq C \left\{ \Theta_{\sigma}(T) + \langle |\psi - \nabla \chi_T| \rangle \right\} \left\{ \|\nabla^2 v_0\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)} \right\} \\ &\quad + C \|f - v_0\|_{C^{\delta}(\partial\Omega)} + C [\Theta_{\sigma}(T)]^{1-\delta} \|\nabla v_0\|_{C^{\delta}(\partial\Omega)}, \end{aligned} \quad (2.15)$$

where $v_0 \in W^{2,2}(\Omega; \mathbb{R}^m) \cap C^{1,\delta}(\overline{\Omega}; \mathbb{R}^m)$ and $\mathcal{L}_0(v_0) = F$ in Ω . The constant C depends only on δ, σ, A , and Ω .

Proof. Let v_{ε} and w_{ε} be defined as in Lemma 2.3. Then

$$\|u_{\varepsilon} - u_0\|_{L^2(\Omega)} \leq \|w_{\varepsilon}\|_{L^2(\Omega)} + \|v_0 - u_0\|_{L^2(\Omega)} + \varepsilon \|\chi_T\|_{\infty} \|\nabla v_0\|_{L^2(\Omega)} + \|v_{\varepsilon}\|_{L^2(\Omega)}. \quad (2.16)$$

In view of (2.9) we only need to handle the last three terms in the right hand side of (2.16).

First, since $\mathcal{L}_0(v_0 - u_0) = 0$ in Ω and Ω is $C^{1,\alpha}$, we obtain

$$\|v_0 - u_0\|_{L^2(\Omega)} \leq C \|v_0 - f\|_{L^2(\partial\Omega)}. \quad (2.17)$$

Next, we note that

$$\varepsilon \|\chi_T\|_{\infty} \|\nabla v_0\|_{L^2(\Omega)} = T^{-1} \|\chi_T\|_{\infty} \|\nabla v_0\|_{L^2(\Omega)} \leq C_{\sigma} \Theta_{\sigma}(T) \|\nabla v_0\|_{L^2(\Omega)}.$$

Finally, recall that $\mathcal{L}_{\varepsilon}(v_{\varepsilon}) = 0$ in Ω and $v_{\varepsilon} = f - v_0 - \varepsilon \chi_T(x/\varepsilon) \nabla v_0$ on $\partial\Omega$. Since Ω is $C^{1,\alpha}$, we may use the Hölder estimates (1.12) to obtain

$$\begin{aligned} \|v_{\varepsilon}\|_{L^2(\Omega)} &\leq C \|v_{\varepsilon}\|_{C^{\delta_1}(\partial\Omega)} \\ &\leq C \|f - v_0\|_{C^{\delta_1}(\partial\Omega)} + C \|\varepsilon \chi_T(x/\varepsilon)\|_{C^{\delta_1}(\partial\Omega)} \|\nabla v_0\|_{C^{\delta_1}(\partial\Omega)} \end{aligned} \quad (2.18)$$

for any $\delta_1 \in (0, 1)$. Note that $\|\varepsilon \chi_T(x/\varepsilon)\|_{\infty} \leq C_{\sigma} \Theta_{\sigma}(T)$ and $\|\varepsilon \chi_T(x/\varepsilon)\|_{C^{0,\delta}} \leq C_{\delta}$. By interpolation this implies that

$$\|\varepsilon \chi_T(x/\varepsilon)\|_{C^{\delta_1}} \leq C [\Theta_{\sigma}(T)]^{1-\delta_2}$$

for any $\sigma \in (0, 1)$ and $0 < \delta_1 < \delta_2 < 1$. As a result, we see that

$$\|v_{\varepsilon}\|_{L^2(\Omega)} \leq C \|f - v_0\|_{C^{\delta}(\partial\Omega)} + C [\Theta_{\sigma}(T)]^{1-\delta} \|\nabla v_0\|_{C^{\delta}(\partial\Omega)}$$

for any $\sigma, \delta \in (0, 1)$. The proof is now complete. \square

Remark 2.5. If we let $v_0 = u_0$ in Lemma 2.4, then

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \omega(\varepsilon) \|u_0\|_{W^{2,p}(\Omega)}, \quad (2.19)$$

where $p > d$ and

$$\omega(\varepsilon) = \omega_\sigma(\varepsilon) = [\Theta_1(\varepsilon^{-1})]^\sigma + \sup_{T \geq \varepsilon^{-1}} \langle |\psi - \nabla \chi_T| \rangle. \quad (2.20)$$

Here we have used the observation $\Theta_\sigma(T) \leq C_\sigma [\Theta_1(T)]^\sigma$ as well as Sobolev imbedding $\|\nabla u_0\|_{C^\delta(\Omega)} \leq C \|u_0\|_{W^{2,p}(\Omega)}$ for $p > d$ and $\delta = 1 - (d/p)$. Note that $\omega(\varepsilon)$ is a nondecreasing continuous function on $(0, 1]$ and $\omega(0+) = 0$.

Estimate (2.19) is one of the main results proved in [21]. In the periodic setting it gives a near optimal convergence rate of $O(\varepsilon^\gamma)$ for any $\gamma \in (0, 1)$. However, since Ω is only assumed to be $C^{1,\alpha}$, the $W^{2,p}$ norm in (2.19) is not convenient in some applications. Our next theorem is an attempt to resolve this issue (see [13] for analogous results in the periodic setting). For simplicity we assume that $F = 0$.

Theorem 2.6. *Suppose that $A(y)$ is uniformly almost-periodic in \mathbb{R}^d and satisfies (1.2). Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d for some $\alpha > 0$. Let u_ε ($\varepsilon \geq 0$) be the weak solution of Dirichlet problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω and $u_\varepsilon = f$ on $\partial\Omega$. Then, for any $\delta \in (0, \alpha)$,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C [\omega(\varepsilon)]^{2/3} \|f\|_{C^{1,\delta}(\partial\Omega)}, \quad (2.21)$$

where $\omega(\varepsilon) = \omega_\sigma(\varepsilon)$ is defined by (2.20) and C depends only on δ , σ , A , and Ω .

Proof. We begin by constructing a family of bounded $C^{1,\alpha}$ domains $\{\Omega_s : s \in (0, 1/2)\}$ such that (1) $\Omega \subset \Omega_s$, (2) for each $s \in (0, 1/2)$, there is a $C^{1,\alpha}$ diffeomorphism $\Lambda_s : \partial\Omega \rightarrow \partial\Omega_s$ with uniform bounds, and (3) $|x - \Lambda_s(x)| \approx \text{dist}(x, \partial\Omega_s) \approx s$ for every $x \in \partial\Omega$. The constants C in the estimates below do not depend on s .

Next, let $f_s(x) = f(\Lambda_s^{-1}(x))$ for $x \in \partial\Omega_s$ and $v = v_s$ be the solution of Dirichlet problem: $\mathcal{L}_0(v) = 0$ in Ω_s and $v = f_s$ on $\partial\Omega_s$. We will show that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \left\{ s + s^{-\frac{1}{2}} \omega(\varepsilon) \right\} \|f\|_{C^{1,\delta}(\partial\Omega)}. \quad (2.22)$$

Estimate (2.21) follows from (2.22) by choosing $s \in (0, 1/2)$ so that $s^{3/2} = c\omega(\varepsilon)$.

To see (2.22), we use Lemma 2.4 to obtain

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \omega(\varepsilon) \{ \|\nabla^2 v\|_{L^2(\Omega)} + \|\nabla v\|_{C^\delta(\Omega)} \} + C \|f - v\|_{C^\delta(\partial\Omega)} \quad (2.23)$$

for any $\delta \in (0, \alpha)$. Since $\mathcal{L}_0(v) = 0$ in Ω_s and Ω_s is $C^{1,\alpha}$,

$$\|f - v\|_{C^\delta(\partial\Omega)} \leq C s \|\nabla v\|_{C^\delta(\Omega_s)} \leq C s \|f_s\|_{C^{1,\delta}(\partial\Omega_s)} \leq C s \|f\|_{C^{1,\delta}(\partial\Omega)}. \quad (2.24)$$

By the interior estimates for \mathcal{L}_0 and the fact that $\Omega \subset \{x \in \Omega_s : \text{dist}(x, \partial\Omega_s) \geq c s\}$, it is not hard to see that

$$\begin{aligned} \int_{\Omega} |\nabla^2 v|^2 dx &\leq C \|\nabla v\|_{L^\infty(\Omega_s)}^2 \int_{\Omega} \frac{dx}{[\text{dist}(x, \partial\Omega_s)]^2} \\ &\leq C s^{-1} \|\nabla v\|_{L^\infty(\Omega_s)}^2 \leq C s^{-1} \|f\|_{C^{1,\delta}(\partial\Omega)}^2. \end{aligned}$$

This, together with (2.23)-(2.24) and the estimate $\|\nabla v\|_{C^\delta(\Omega)} \leq C \|f\|_{C^{1,\delta}(\partial\Omega)}$, yields (2.22). \square

2.4 Convergence rates: Neumann boundary conditions

In this subsection we establish estimates on convergence rates for the Neumann problem (1.6) under an additional assumption that

$$\sup_{T \geq 1} \|\nabla \chi_T\|_{L^\infty(\mathbb{R}^d)} \leq C_0 < \infty. \quad (2.25)$$

This condition follows from the uniform interior Lipschitz estimates (see Remark 4.7). In particular, it holds under the assumptions on A in Theorem 1.1.

Lemma 2.7. *Suppose that A is uniformly almost-periodic and satisfies (1.2). Also assume that the condition (2.25) holds. Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Let*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_0(v_0) = F & \text{in } \Omega, \\ \frac{\partial v_0}{\partial \nu_0} = g_0 & \text{on } \partial\Omega, \end{cases} \quad (2.26)$$

where $F \in L^2(\Omega; \mathbb{R}^m)$ and $g, g_0 \in L^2(\partial\Omega; \mathbb{R}^m)$. Suppose $\int_{\Omega} u_\varepsilon = \int_{\Omega} v_0$. Then, for any $\sigma \in (0, 1)$,

$$\begin{aligned} \|u_\varepsilon - v_0\|_{L^2(\Omega)} &\leq C \|g - g_0\|_{H^{-1/2}(\partial\Omega)} + C \left\{ \Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle \right\} \|\nabla^2 v_0\|_{L^2(\Omega)} \\ &\quad + C \left\{ [\Theta_\sigma(T)]^{\frac{1}{2}} + \langle |\psi - \nabla \chi_T| \rangle \right\} \|(\nabla v_0)^*\|_{L^2(\partial\Omega)}, \end{aligned} \quad (2.27)$$

where $T = \varepsilon^{-1}$ and $(\nabla v_0)^*$ denotes the non-tangential maximal function of ∇v_0 . The constant C in (2.27) depends only on σ , A , and Ω .

Proof. As in the case of Dirichlet boundary condition, we consider

$$w_\varepsilon = u_\varepsilon - v_0 - \varepsilon \chi_T(x/\varepsilon) \nabla v_0,$$

where $T = \varepsilon^{-1}$. We will show that

$$\begin{aligned} \|\nabla w_\varepsilon\|_{L^2(\Omega)} &\leq C \|g - g_0\|_{H^{-1/2}(\partial\Omega)} + C \left\{ \Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle \right\} \|\nabla^2 v_0\|_{L^2(\Omega)} \\ &\quad + C \left\{ [\Theta_\sigma(T)]^{\frac{1}{2}} + \langle |\psi - \nabla \chi_T| \rangle \right\} \|(\nabla v_0)^*\|_{L^2(\partial\Omega)}. \end{aligned} \quad (2.28)$$

Since $|\int_{\Omega} w_{\varepsilon}| \leq \varepsilon \|\chi_T\|_{\infty} \|\nabla v_0\|_{L^1(\Omega)} \leq C \Theta_{\sigma}(T) \|\nabla v_0\|_{L^1(\Omega)}$, the estimate (2.27) follows from (2.28) by Poincaré inequality.

To prove (2.28), we observe that

$$\begin{aligned} & \int_{\Omega} \nabla w_{\varepsilon} \cdot A(x/\varepsilon) \nabla w_{\varepsilon} dx \\ &= \langle w_{\varepsilon}, g - g_0 \rangle - \int_{\Omega} \nabla w_{\varepsilon} \cdot B_T(x/\varepsilon) \nabla v_0 - \int_{\Omega} \nabla w_{\varepsilon} \cdot A(x/\varepsilon) \varepsilon \chi_T(x/\varepsilon) \nabla^2 v_0, \end{aligned} \quad (2.29)$$

where $B_T(y) = \widehat{A} - A(y) - A(y) \nabla \chi_T(y)$, and we have used the fact

$$\int_{\Omega} \nabla w_{\varepsilon} \cdot (A(x/\varepsilon) \nabla u_{\varepsilon} - \widehat{A} \nabla v_0) = \langle w_{\varepsilon}, g - g_0 \rangle.$$

Since $\int_{\partial\Omega} (g - g_0) = 0$,

$$\begin{aligned} |\langle w_{\varepsilon}, g - g_0 \rangle| &\leq \|g - g_0\|_{H^{-1/2}(\partial\Omega)} \|w_{\varepsilon} - E\|_{H^{1/2}(\partial\Omega)} \\ &\leq C \|g - g_0\|_{H^{-1/2}(\partial\Omega)} \|\nabla w_{\varepsilon}\|_{L^2(\Omega)}, \end{aligned}$$

where $E = \int_{\Omega} w_{\varepsilon}$. Also, the last term in the right hand of (2.29) is bounded by

$$C_{\sigma} \Theta_{\sigma}(T) \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \|\nabla^2 v_0\|_{L^2(\Omega)}.$$

Furthermore, since $|\langle B_T \rangle| \leq C \langle |\psi - \nabla \chi_T| \rangle$, in view of (2.29), it suffices to show that

$$\begin{aligned} & \left| \int_{\Omega} \nabla w_{\varepsilon} \cdot \{B_T(x/\varepsilon) - \langle B_T \rangle\} \nabla v_0 \right| \\ & \leq C \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \{\Theta_{\sigma}(T) + \langle |\psi - \nabla \chi_T| \rangle\} \|\nabla^2 v_0\|_{L^2(\Omega)} \\ & \quad + C \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} [\Theta_{\sigma}(T)]^{\frac{1}{2}} \|(\nabla v_0)^*\|_{L^2(\partial\Omega)}. \end{aligned} \quad (2.30)$$

This will be done by using a line of argument similar to that used in the proof of Theorem 7.3 in [21] as well as in the proof of Lemma 2.3.

Let $H = H_T \in W_{loc}^{2,2}(\mathbb{R})$ be a solution of (2.12) that satisfies (2.13)-(2.14). In view of the first estimate in (2.13), it suffices to prove (2.30) with $B_T(x/\varepsilon) - \langle B_T \rangle$ replaced by $\Delta H(x/\varepsilon)$. Let $\varphi = \varphi_{\delta} \in C_0^{\infty}(\mathbb{R}^d)$ be a cut-off function such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $\text{dist}(x, \partial\Omega) \geq 2c\delta$, $\varphi(x) = 0$ if $\text{dist}(x, \partial\Omega) \leq c\delta$, and $|\nabla \varphi| \leq C\delta^{-1}$, where $\delta \in (0, 1)$ is to be determined. A direct computation shows that for each $1 \leq j \leq d$ and $1 \leq \beta \leq m$,

$$\begin{aligned} \frac{\partial w_{\varepsilon}^{\alpha}}{\partial x_i} \cdot \Delta h_{ij}^{\alpha\beta}(x/\varepsilon) &= \frac{\partial}{\partial x_k} \left\{ \frac{\partial w_{\varepsilon}^{\alpha}}{\partial x_i} \cdot \varepsilon \frac{\partial h_{ij}^{\alpha\beta}}{\partial x_k}(x/\varepsilon) \right\} - \frac{\partial}{\partial x_i} \left\{ \frac{\partial w_{\varepsilon}^{\alpha}}{\partial x_k} \cdot \varepsilon \frac{\partial h_{ij}^{\alpha\beta}}{\partial x_k}(x/\varepsilon) \right\} \\ &\quad + \frac{\partial w_{\varepsilon}^{\alpha}}{\partial x_k} \cdot \frac{\partial^2 h_{ij}^{\alpha\beta}}{\partial x_i \partial x_k}(x/\varepsilon), \end{aligned}$$

where the summation convention is used. It follows that

$$\begin{aligned}
\left| \int_{\Omega} \nabla w_{\varepsilon} \cdot \Delta H(x/\varepsilon) (\nabla v_0) \varphi \right| &\leq C \varepsilon \int_{\Omega} |\nabla w_{\varepsilon}| |\nabla H(x/\varepsilon)| |\nabla((\nabla v_0) \varphi)| \\
&\quad + C \int_{\Omega} \left| \frac{\partial w_{\varepsilon}^{\alpha}}{\partial x_k} \right| \left| \frac{\partial h_{ij}^{\alpha\beta}}{\partial x_i \partial x_k}(x/\varepsilon) \right| \left| \frac{\partial v_0^{\beta}}{\partial x_j} \right| \varphi \\
&\leq C \Theta_{\sigma}(T) \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \left\{ \|\nabla^2 v_0\|_{L^2(\Omega)} + \delta^{-1/2} \|(\nabla v_0)^*\|_{L^2(\partial\Omega)} \right\},
\end{aligned} \tag{2.31}$$

where we have used estimates (2.13) and (2.14) as well as the observation

$$\|(\nabla v_0)(\nabla \varphi)\|_{L^2(\Omega)} \leq C \delta^{-1/2} \|(\nabla v_0)^*\|_{L^2(\partial\Omega)}.$$

Finally, using the condition (2.25), we see that

$$\|\Delta H\|_{\infty} \leq T^{-2} \|H\|_{\infty} + 2 \|B_T\|_{\infty} \leq C + C \|\nabla \chi_T\|_{\infty} \leq C.$$

Hence,

$$\begin{aligned}
\left| \int_{\Omega} \nabla w_{\varepsilon} \cdot \Delta H(x/\varepsilon) (\nabla v_0) (1 - \varphi) \right| &\leq C \|\Delta H\|_{\infty} \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \left\{ \int_{\substack{x \in \Omega \\ \text{dist}(x, \partial\Omega) \leq 2c\delta}} |\nabla v_0|^2 \right\}^{1/2} \\
&\leq C \delta^{1/2} \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \|(\nabla v_0)^*\|_{L^2(\partial\Omega)}.
\end{aligned}$$

By choosing $\delta = c \Theta_{\sigma}(T)$, this, together with (2.31), completes the proof. \square

Remark 2.8. Let u_{ε} ($\varepsilon \geq 0$) be the weak solution of (1.6) in a bounded Lipschitz domain Ω . It follows from Lemma 2.7 that

$$\begin{aligned}
\|u_{\varepsilon} - u_0\|_{L^2(\Omega)} &\leq C \left\{ \Theta_{\sigma}(T) + \langle |\psi - \nabla \chi_T| \rangle \right\} \|\nabla^2 u_0\|_{L^2(\Omega)} \\
&\quad + C \left\{ [\Theta_{\sigma}(T)]^{\frac{1}{2}} + \langle |\psi - \nabla \chi_T| \rangle \right\} \|(\nabla u_0)^*\|_{L^2(\partial\Omega)}.
\end{aligned} \tag{2.32}$$

This estimate is not sharp in the periodic setting. It only gives $\|u_{\varepsilon} - u_0\|_{L^2(\Omega)} = O(\varepsilon^{\gamma})$ for any $0 < \gamma < (1/2)$.

Theorem 2.9. Suppose that A satisfies the same condition as in Lemma 2.7. Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d with connected boundary for some $\alpha > 0$. Let

$$\begin{cases} \mathcal{L}_{\varepsilon}(u_{\varepsilon}) = 0 & \text{in } \Omega, \\ \frac{\partial u_{\varepsilon}}{\partial \nu_{\varepsilon}} = g & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_0(u_0) = 0 & \text{in } \Omega, \\ \frac{\partial u_0}{\partial \nu_0} = g & \text{on } \partial\Omega, \end{cases} \tag{2.33}$$

where $g \in L^2(\partial\Omega; \mathbb{R}^m)$. Assume $\int_{\Omega} u_{\varepsilon} = \int_{\Omega} u_0$. Then

$$\|u_{\varepsilon} - u_0\|_{L^2(\Omega)} \leq C \left\{ \Theta_{\sigma}(T) + \langle |\psi - \nabla \chi_T| \rangle \right\}^{1/2} \|g\|_{L^2(\partial\Omega)}, \tag{2.34}$$

where $T = \varepsilon^{-1}$, $\sigma \in (0, 1)$, and C depends only on σ , A , and Ω .

Proof. We begin by constructing a family of $C^{1,\alpha}$ domains $\{\Omega_t : t \in (0, 1)\}$ with the property that (1) $\Omega \subset \Omega_t$, (2) there exist $C^{1,\alpha}$ diffeomorphisms $\Lambda_t : \partial\Omega \rightarrow \partial\Omega_t$ with uniform bounds such that $\text{dist}(x, \Lambda_t(x)) \approx \text{dist}(x, \partial\Omega_t) \approx t$ for any $x \in \partial\Omega$. Let $v = v_t$ be the weak solution to Dirichlet problem: $\mathcal{L}_0(v) = 0$ in Ω_t and $v = f_t$ on $\partial\Omega$, where $f_t(x) = u_0(\Lambda_t^{-1}(x))$ for $x \in \partial\Omega_t$.

Next we use the non-tangential maximal function estimates

$$\begin{cases} \|(w)^*\|_{L^2(\partial\Omega_t)} \leq C \|w\|_{L^2(\partial\Omega_t)}, & \|(\nabla w)^*\|_{L^2(\partial\Omega_t)} \leq C \|\nabla_{\tan} w\|_{L^2(\partial\Omega_t)}, \\ \|(\nabla w)^*\|_{L^2(\partial\Omega_t)} \leq C \left\| \frac{\partial w}{\partial \nu_0} \right\|_{L^2(\partial\Omega_t)} \end{cases}$$

for the L^2 Dirichlet and Neumann problems for the system $\mathcal{L}_0(w) = 0$ in $C^{1,\alpha}$ domains to control $\|u_0 - v\|_{L^2(\Omega)}$. This gives

$$\begin{aligned} \|u_0 - v\|_{L^2(\Omega)} &\leq C \|u_0 - v\|_{L^2(\partial\Omega)} \leq C t \|(\nabla v)^*\|_{L^2(\partial\Omega_t)} \\ &\leq C t \|\nabla_{\tan} v\|_{L^2(\partial\Omega_t)} \leq C t \|\nabla u_0\|_{L^2(\partial\Omega)} \\ &\leq C t \|g\|_{L^2(\partial\Omega)}. \end{aligned} \tag{2.35}$$

To handle $\|u_\varepsilon - v\|_{L^2(\Omega)}$, we use Lemma 2.7 to obtain

$$\begin{aligned} \|u_\varepsilon - v\|_{L^2(\Omega)} &\leq C \|g - g_0\|_{H^{-1/2}(\partial\Omega)} + C \{\Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle\} \|\nabla^2 v\|_{L^2(\Omega)} \\ &\quad + C \left\{ [\Theta_\sigma(T)]^{1/2} + \langle |\psi - \nabla \chi_T| \rangle \right\} \|(\nabla v)^*\|_{L^2(\partial\Omega)}, \end{aligned} \tag{2.36}$$

where $g_0 = \frac{\partial v}{\partial \nu_0}$. Since $\mathcal{L}_0(u_0 - v) = 0$ in Ω , we see that

$$\begin{aligned} \|g - g_0\|_{H^{-1/2}(\partial\Omega)} &\leq C \|u_0 - v\|_{H^1(\Omega)} \leq C \|u_0 - v\|_{H^{1/2}(\partial\Omega)} \\ &\leq C \|u_0 - v\|_{L^2(\partial\Omega)}^{1/2} \|u_0 - v\|_{H^1(\partial\Omega)}^{1/2} \\ &\leq C t^{1/2} \|(\nabla v)^*\|_{L^2(\partial\Omega)} \leq C t^{1/2} \|g\|_{L^2(\partial\Omega)}. \end{aligned} \tag{2.37}$$

Also, since $\mathcal{L}_0(v) = 0$ in Ω_t , by the square function estimate [7],

$$\left\{ \int_{\Omega_t} |\nabla^2 v(x)|^2 \text{dist}(x, \partial\Omega_t) dx \right\}^{1/2} \leq C \|v\|_{H^1(\partial\Omega_t)},$$

we obtain

$$\|\nabla^2 v\|_{L^2(\Omega)} \leq C t^{-1/2} \|g\|_{L^2(\partial\Omega)}. \tag{2.38}$$

In view of (2.35)-(2.38) we have proved that

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(\Omega)} &\leq C t^{-1/2} \{\Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle\} \|g\|_{L^2(\partial\Omega)} \\ &\quad + C \left\{ [\Theta_\sigma(T)]^{1/2} + \langle |\psi - \nabla \chi_T| \rangle \right\} \|g\|_{L^2(\partial\Omega)} + C t^{1/2} \|g\|_{L^2(\partial\Omega)}. \end{aligned}$$

Finally, the estimate (2.34) follows by choosing $t = c\{\Theta_\sigma(T) + \langle |\psi - \nabla \chi_T| \rangle\}$. \square

3 A general scheme for Lipschitz estimates at large scale

In this section we present a general scheme for proving Lipschitz estimates at large scale in homogenization. As we pointed out in Introduction, the scheme, which was motivated by the compactness method in [2], was recently formulated by the first author and C. Smart in [1]. The L^2 version of the scheme in this section is a slight variation of the one given in [1].

Lemma 3.1. *Let $\{F_0, F_1, \dots, F_\ell\}$ and $\{p_0, p_1, \dots, p_\ell\}$ be two sequences of nonnegative numbers. Suppose that for $0 \leq j \leq \ell - 1$,*

$$p_{j+1} \leq p_j + C_0 \max \{F_j, F_{j+1}\}, \quad (3.1)$$

and for $1 \leq j \leq \ell - 1$,

$$F_{j+1} \leq \frac{1}{2}F_j + \eta_j K + \eta_j \max \{p_0, \dots, p_{j-1}\} + \eta_j \max \{F_0, \dots, F_{j-1}\}, \quad (3.2)$$

where $K \geq 0$, $0 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_{\ell-1} = \eta_\ell$ and $\eta_1 + \eta_2 + \dots + \eta_\ell \leq C_1$. Then for $1 \leq j \leq \ell$,

$$p_j \leq C (K + p_0 + F_0 + F_1), \quad (3.3)$$

$$F_j \leq C (2^{-j} + \eta_j)(K + p_0 + F_0 + F_1), \quad (3.4)$$

where C depends only on C_0 and C_1 .

Proof. The proof of this lemma is essentially contained in the proof of [1, Lemma 5.1]. We provide a proof here for the sake of completeness.

By considering $\tilde{p}_j = p_j + K$, we may assume that $K = 0$. Let

$$T_j = F_j - 2\eta_j \max \{p_0, \dots, p_{j-1}\} - 2\eta_j \max \{F_0, \dots, F_{j-1}\}. \quad (3.5)$$

Note that

$$\begin{aligned} T_{j+1} &= F_{j+1} - 2\eta_{j+1} \max \{p_0, \dots, p_j\} - 2\eta_{j+1} \max \{F_0, \dots, F_j\} \\ &\leq \frac{1}{2}F_j + \eta_j \max \{p_0, \dots, p_{j-1}\} + \eta_j \max \{F_0, \dots, F_{j-1}\} \\ &\quad - 2\eta_{j+1} \max \{p_0, \dots, p_j\} - 2\eta_{j+1} \max \{F_0, \dots, F_j\} \\ &\leq \frac{1}{2}F_j + (\eta_j - 2\eta_{j+1}) \max \{p_0, \dots, p_{j-1}\} + (\eta_j - 2\eta_{j+1}) \max \{F_0, \dots, F_{j-1}\}, \end{aligned}$$

where we have used (3.2) for the first inequality. Since $\eta_j - 2\eta_{j+1} \leq -\eta_j$, we obtain $T_{j+1} \leq (1/2)T_j$ for $1 \leq j \leq \ell - 1$. It follows that $T_j \leq (1/2)^{j-1}T_1 \leq (1/2)^{j-1}F_1$. Hence,

$$F_j \leq (1/2)^{j-1}F_1 + 2\eta_j \max \{p_0, \dots, p_{j-1}\} + 2\eta_j \max \{F_0, \dots, F_{j-1}\}. \quad (3.6)$$

Next we will show that for $0 \leq j \leq \ell$,

$$F_j \leq C_2 \left\{ (2^{-j} + \eta_j)(F_0 + F_1) + \eta_j \max \{p_0, \dots, p_{j-1}\} \right\}, \quad (3.7)$$

where C_2 depends only on C_1 . To prove (3.7), we claim that for $1 \leq j \leq \ell$,

$$F_j \leq 2(1 + 2\eta_1) \cdots (1 + 2\eta_j) \left\{ (2^{-j} + \eta_j)(F_0 + F_1) + \eta_j \max \{p_0, \dots, p_{j-1}\} \right\}. \quad (3.8)$$

Since $\eta_1 + \eta_2 + \cdots + \eta_\ell \leq C_1$, one may use the inequality $\ln(1+x) \leq x$ for $x \geq 0$ to see that

$$(1 + C\eta_1) \cdots (1 + C\eta_\ell) \leq e^{CC_1}. \quad (3.9)$$

As a result, estimate (3.7) follows from (3.8).

Estimate (3.8) is proved by induction, using (3.6). Indeed, suppose (3.8) holds for $1 \leq j \leq i$. Then

$$\max \{F_0, \dots, F_i\} \leq 2(1 + 2\eta_1) \cdots (1 + 2\eta_i) \left\{ \left(\frac{1}{2} + \eta_i\right)(F_0 + F_1) + \eta_i \max \{p_0, \dots, p_{i-1}\} \right\},$$

where we have used the monotonicity of η_j . This, together with (3.6), gives

$$\begin{aligned} F_{i+1} &\leq (1/2)^i F_1 + 2\eta_{i+1} \max \{p_0, \dots, p_i\} + 2\eta_{i+1} \max \{F_0, \dots, F_i\} \\ &\leq (1/2)^i F_1 + 2\eta_{i+1} \max \{p_0, \dots, p_i\} \\ &\quad + 2\eta_{i+1} \cdot 2(1 + 2\eta_1) \cdots (1 + 2\eta_i) \left(\frac{1}{2} + \eta_i\right)(F_0 + F_1) \\ &\quad + 2\eta_{i+1} \cdot 2(1 + 2\eta_1) \cdots (1 + 2\eta_i) \cdot \eta_i \max \{p_0, \dots, p_{i-1}\} \\ &\leq 2(1 + 2\eta_1) \cdots (1 + 2\eta_{i+1}) \left\{ (2^{-i-1} + \eta_{i+1})(F_0 + F_1) + \eta_{i+1} \max \{p_0, \dots, p_i\} \right\}. \end{aligned}$$

Finally, we give the proof for estimate (3.3), which, together with (3.7), yields (3.4). To this end we use (3.1) and (3.7) to obtain

$$\begin{aligned} p_{j+1} &\leq p_j + C_0 \max \{F_j, F_{j+1}\} \\ &\leq p_j + C (2^{-j-1} + \eta_{j+1})(F_0 + F_1) + C \eta_{j+1} \max \{p_0, \dots, p_j\} \\ &\leq (1 + C\eta_{j+1}) \max \{p_0, \dots, p_j\} + C (2^{-j-1} + \eta_{j+1})(F_0 + F_1), \end{aligned}$$

where C depends only on C_0 and C_1 . By a simple induction argument it follows

$$p_j \leq (1 + C\eta_1) \cdots (1 + C\eta_j) \left\{ p_0 + F_0 + F_1 + C \sum_{k=1}^j (2^{-k} + \eta_k)(F_0 + F_1) \right\}, \quad (3.10)$$

where C depends only on C_0 and C_1 . In view of (3.9) this gives the desired estimate (3.3). The proof is now complete. \square

Theorem 3.2. Let $B_r = B(0, r)$ and $u \in L^2(B_1; \mathbb{R}^m)$. Let $0 < \varepsilon < 1/4$. Suppose that for each $r \in (\varepsilon, 1/4)$, there exists $w = w_r \in L^2(B_r; \mathbb{R}^m)$ such that

$$\left\{ \int_{B_r} |u - w|^2 \right\}^{1/2} \leq \eta(\varepsilon/r) \left\{ \inf_{q \in \mathbb{R}^m} \left(\int_{B_{2r}} |u - q|^2 \right)^{1/2} + r K \right\}, \quad (3.11)$$

and

$$\begin{aligned} \frac{1}{\theta} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_{\theta r}} |w(x) - Mx - q|^2 dx \right)^{1/2} \\ \leq \frac{1}{2} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_r} |w(x) - Mx - q|^2 dx \right)^{1/2}, \end{aligned} \quad (3.12)$$

where $K \geq 0$, $\theta \in (0, 1/4)$, and $\eta(t)$ is a nondecreasing function on $(0, 1]$. Assume that

$$I = \int_0^1 \frac{\eta(t)}{t} dt < \infty. \quad (3.13)$$

Then, for $\varepsilon < t < (1/4)$,

$$\frac{1}{t} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_t} |u - q|^2 \right\}^{1/2} \leq C \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\}, \quad (3.14)$$

and

$$\begin{aligned} \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B_t} |u(x) - Mx - q|^2 dx \right\}^{1/2} \\ \leq C \{t^\alpha + \eta(\varepsilon/t)\} \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\}, \end{aligned} \quad (3.15)$$

where $\alpha = \alpha(\theta) > 0$ and C depends only on d , m , θ , and I .

Proof. It follows from the assumptions (3.12) and (3.11) that for $r \in (\varepsilon, 1/2)$,

$$\begin{aligned} \frac{1}{\theta r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_{\theta r}} |u - Mx - q|^2 dx \right)^{1/2} \\ \leq \frac{C}{r} \left\{ \int_{B_r} |u - w_r|^2 \right\}^{1/2} + \frac{1}{2r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_r} |u - Mx - q|^2 dx \right)^{1/2} \\ \leq C \eta(\varepsilon/r) \left\{ \frac{1}{2r} \inf_{q \in \mathbb{R}^m} \left(\int_{B_{2r}} |u - q|^2 \right)^{1/2} + K \right\} \\ + \frac{1}{2r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_r} |u - Mx - q|^2 dx \right)^{1/2}. \end{aligned} \quad (3.16)$$

Let $r_j = \theta^{j+1}$ for $0 \leq j \leq \ell$, where ℓ is chosen so that $\theta^{\ell+2} < \varepsilon \leq \theta^{\ell+1}$ (we may assume that $\varepsilon < \theta$). Let

$$\begin{aligned} F_j &= \frac{1}{r_j} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B_{r_j}} |u - Mx - q|^2 dx \right)^{1/2} \\ &= \frac{1}{r_j} \inf_{q \in \mathbb{R}^m} \left(\int_{B_{r_j}} |u - M_j x - q|^2 dx \right)^{1/2} \end{aligned} \quad (3.17)$$

and $p_j = |M_j|$. Note that by (3.16),

$$\begin{aligned} F_{j+1} &\leq \frac{1}{2} F_j + C \eta(\varepsilon \theta^{-j-1}) \left\{ \frac{1}{2r_j} \inf_{q \in \mathbb{R}^m} \left(\int_{B_{2r_j}} |u - q|^2 \right)^{1/2} + K \right\} \\ &\leq \frac{1}{2} F_j + C \eta(\varepsilon \theta^{-j-1}) \{K + F_{j-1} + p_{j-1}\}. \end{aligned} \quad (3.18)$$

Also observe that

$$\begin{aligned} |M_{j+1} - M_j| &\leq \frac{C}{r_{j+1}} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_{r_{j+1}}} |(M_{j+1} - M_j)x - q|^2 dx \right\}^{1/2} \\ &\leq \frac{C}{r_{j+1}} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_{r_{j+1}}} |u - M_{j+1}x - q|^2 \right\}^{1/2} + \frac{C}{r_{j+1}} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_{r_{j+1}}} |u - M_j x - q|^2 \right\}^{1/2} \\ &\leq C (F_j + F_{j+1}). \end{aligned}$$

This gives

$$p_{j+1} = |M_{j+1}| \leq |M_j| + C (F_j + F_{j+1}) = p_j + C (F_j + F_{j+1}). \quad (3.19)$$

We further note that

$$\sum_{j=0}^{\ell-1} \eta(\varepsilon \theta^{-j-1}) \leq \frac{1}{\ln(1/\theta)} \int_0^1 \frac{\eta(t)}{t} dt < \infty.$$

Thus the sequences $\{F_0, F_1, \dots, F_\ell\}$ and $\{p_0, p_1, \dots, p_\ell\}$ satisfy the conditions in Lemma 3.1. Consequently, we obtain

$$\begin{aligned} F_j &\leq C (2^{-j} + \eta(\varepsilon \theta^{-j-1})) (K + p_0 + F_0 + F_1) \\ &\leq C (2^{-j} + \eta(\varepsilon \theta^{-j-1})) \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\}, \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} p_j &\leq C (K + p_0 + F_0 + F_1) \\ &\leq C \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\}. \end{aligned} \quad (3.21)$$

Finally, given any $t \in (\varepsilon, \theta)$ (the case $t \geq \theta$ is trivial), we choose $j \geq 0$ so that $\theta^{j+2} < t \leq \theta^{j+1}$. Then

$$\begin{aligned} \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B_t} |u - Mx - q|^2 \right\}^{1/2} &\leq C F_j \\ &\leq C \{2^{-j} + \eta(\varepsilon \theta^{-j-1})\} \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\} \\ &\leq C \{t^\alpha + \eta(\varepsilon/t)\} \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\}, \end{aligned}$$

where $\alpha = \alpha(\theta) > 0$, and

$$\begin{aligned} \frac{1}{t} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_t} |u - q|^2 \right\}^{1/2} &\leq \frac{C}{r_j} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_{r_j}} |u - q|^2 \right\}^{1/2} \\ &\leq C \{F_j + p_j\} \\ &\leq C \left\{ K + \left(\int_{B_1} |u|^2 \right)^{1/2} \right\}. \end{aligned}$$

This completes the proof. \square

Remark 3.3. The L^2 norm plays no role in the proof above. Theorem 3.2 continues to hold if one replaces the L^2 average over B_r by the L^p average over B_r for any $1 \leq p < \infty$ or by the L^∞ norm over B_r .

In the next section we will use Theorem 3.2 to establish uniform interior Lipschitz estimates for \mathcal{L}_ε . The function $w = w_r(x)$ will be a suitably chosen solution of $\mathcal{L}_0(w) = 0$ in B_r . Since the homogenized operator \mathcal{L}_0 has constant coefficients, its solutions possess $C^{1,\alpha}$ estimates that make (3.12) possible. As we shall see in Sections 7 and 8, with our results on convergence rates in Section 2, this approach for the interior Lipschitz estimates may be adapted for boundary Lipschitz estimates with either Dirichlet or Neumann conditions.

4 Interior Lipschitz estimates

In this section we establish the uniform Lipschitz estimates for $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$. Our approach is based on Theorem 3.2. The key ingredients are provided by the next three lemmas.

Lemma 4.1. *Let $B_r = B(0, r)$. Suppose that $u_\varepsilon \in H^1(B_{2r}; \mathbb{R}^m)$ and $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_{2r} for some $0 < \varepsilon < r < 1$. Then there exists $w \in H^1(B_r; \mathbb{R}^m)$ such that $\mathcal{L}_0(w) = 0$ in B_r and*

$$\left\{ \int_{B_r} |u_\varepsilon - w|^2 \right\}^{1/2} \leq C_\delta [\omega(\varepsilon/r)]^{\frac{2}{3}-\delta} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_{2r}} |u_\varepsilon - q|^2 \right\}^{1/2} \quad (4.1)$$

for any $\delta \in (0, 1/4)$, where $\omega(t) = \omega_\sigma(t)$ is defined by (2.20). The constant C_δ depends only on δ , σ , and A .

Proof. By a simple rescaling we may assume that $r = 1$. By subtracting a constant we may also assume $\int_{B_2} u_\varepsilon = 0$. Let $f_t = u_\varepsilon * \varphi_t$, where $\varphi_t(x) = t^{-d} \varphi(x/t)$, $\varphi \in C_0^\infty(B_1)$ and $\int_{\mathbb{R}^d} \varphi = 1$. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in B_2 , using the interior Hölder estimate for \mathcal{L}_ε ,

$$\|u_\varepsilon\|_{C^\beta(B_{7/4})} \leq C_\beta \|u_\varepsilon\|_{L^2(B_2)} \quad \text{for any } 0 < \beta < 1$$

(see Theorem 3.4 in [21]), it is easy to see that

$$\|f_t - u_\varepsilon\|_{C^\alpha(B_{3/2})} \leq C_{\alpha,\beta} t^{\beta-\alpha} \|u_\varepsilon\|_{L^2(B_2)} \quad (4.2)$$

and

$$\|f_t\|_{C^{1,\alpha}(B_{3/2})} \leq C_{\alpha,\beta} t^{\beta-\alpha-1} \|u_\varepsilon\|_{L^2(B_2)}, \quad (4.3)$$

where $t \in (0, 1/4)$ and $0 < \alpha < \beta < 1$. We now solve the Dirichlet problems

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = 0 & \text{in } B_{5/4}, \\ v_\varepsilon = f_t & \text{on } \partial B_{5/4}, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_0(w) = 0 & \text{in } B_{5/4}, \\ w = f_t & \text{on } \partial B_{5/4}, \end{cases} \quad (4.4)$$

where $t \in (0, 1/4)$ is to be determined. Since $\mathcal{L}_\varepsilon(u_\varepsilon - v_\varepsilon) = 0$ in $B_{5/4}$, it follows from (1.12) and (4.2) that

$$\|u_\varepsilon - v_\varepsilon\|_{L^\infty(B_{5/4})} \leq C_\alpha \|u_\varepsilon - f_t\|_{C^\alpha(\partial B_{5/4})} \leq C_{\alpha,\beta} t^{\beta-\alpha} \|u_\varepsilon\|_{L^2(B_2)}. \quad (4.5)$$

Also, observe that by Theorem 2.6,

$$\begin{aligned} \|v_\varepsilon - w\|_{L^2(B_{5/4})} &\leq C_\alpha [\omega(\varepsilon)]^{2/3} \|f_t\|_{C^{1,\alpha}(\partial B_{5/4})} \\ &\leq C_{\alpha,\beta} t^{\beta-\alpha-1} [\omega(\varepsilon)]^{2/3} \|u_\varepsilon\|_{L^2(B_2)}. \end{aligned} \quad (4.6)$$

In view of (4.5) and (4.6) we obtain

$$\begin{aligned} \|u_\varepsilon - w\|_{L^2(B_1)} &\leq \|u_\varepsilon - v_\varepsilon\|_{L^2(B_1)} + \|v_\varepsilon - w\|_{L^2(B_1)} \\ &\leq C_{\alpha,\beta} t^{\beta-\alpha} \{1 + t^{-1} [\omega(\varepsilon)]^{2/3}\} \|u_\varepsilon\|_{L^2(B_2)}. \end{aligned} \quad (4.7)$$

We now choose $t = c [\eta(\varepsilon)]^{2/3} \in (0, 1/4)$, $\alpha = (3/4)\delta$, and $\beta = 1 - \alpha$, where $\delta \in (0, 1/4)$. This gives

$$\begin{aligned} \|u_\varepsilon - w\|_{L^2(B_1)} &\leq C_\delta [\omega(\varepsilon)]^{\frac{2}{3}-\delta} \|u_\varepsilon\|_{L^2(B_2)} \\ &\leq C_\delta [\omega(\varepsilon)]^{\frac{2}{3}-\delta} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B_2} |u_\varepsilon - q|^2 \right\}^{1/2}, \end{aligned} \quad (4.8)$$

where we have used the fact $\int_{B_2} u_\varepsilon = 0$ for the last inequality. \square

Lemma 4.2. Suppose $w \in H^1(B_r; \mathbb{R}^m)$ and $\mathcal{L}_0(w) = 0$ in B_r , where $B_r = B(0, r)$. Then, for any $\theta \in (0, 1/2)$,

$$\frac{1}{\theta} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B_{\theta r}} |w - Mx - q|^2 \right\}^{1/2} \leq C\theta \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B_r} |w - Mx - q|^2 \right\}^{1/2}, \quad (4.9)$$

where C depends only on d , m , and μ . As a result, by choosing θ so small that $C\theta < (1/2)$, solutions of $\mathcal{L}_0(w) = 0$ in B_r satisfy the condition (3.12) in Theorem 3.2.

Proof. Estimate (4.9) follows readily from the interior C^2 estimates for \mathcal{L}_0 . Indeed, by rescaling, we may assume that $r = 1$. In this case the left hand side of (4.9) is bounded by $C\theta \|\nabla^2 w\|_{L^\infty(B_\theta)}$. Since $\mathcal{L}_0(w - Mx - q) = 0$ in B_1 for any $M \in \mathbb{R}^{m \times d}$ and $q \in \mathbb{R}^m$, by the C^2 estimates for \mathcal{L}_0 ,

$$\theta \|\nabla^2 w\|_{L^\infty(B_\theta)} \leq \theta \|\nabla^2 w\|_{L^\infty(B_{1/2})} \leq C\theta \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B_1} |w - Mx - q|^2 \right\}^{1/2}, \quad (4.10)$$

where C depends only on d , m , and μ . The proof is complete. \square

Lemma 4.3. Suppose that there exist $C_0 > 0$ and $N > (5/2)$ such that

$$\rho(R) \leq C_0 [\log R]^{-N} \quad \text{for any } R \geq 2,$$

where $\rho(R)$ is defined by (1.4). Then there exist $\sigma \in (0, 1)$ and $\delta \in (0, 1/4)$ such that

$$\int_0^1 [\omega_\sigma(t)]^{\frac{2}{3}-\delta} \frac{dt}{t} < \infty,$$

where $\omega_\sigma(t)$ is defined by (2.20).

Proof. It follows from the definition of $\Theta_\sigma(T)$ that

$$\Theta_\sigma(T) \leq \rho(\sqrt{T}) + \left(\frac{1}{\sqrt{T}} \right)^\sigma \leq C_\sigma [\log T]^{-N}$$

for $T \geq 2$. Also, it was proved in [21, Theorem 6.6] that

$$\langle |\psi - \nabla \chi_T| \rangle \leq C_\sigma \int_{T/2}^\infty \frac{\Theta_\sigma(r)}{r} dr$$

for any $\sigma \in (0, 1)$. As a result, if $\sigma = 1 - N^{-1}$, we obtain

$$\begin{aligned} \eta_\sigma(t) &= [\Theta_1(t^{-1})]^\sigma + \sup_{T \geq t^{-1}} \langle |\psi - \nabla \chi_T| \rangle \\ &\leq [\Theta_1(t^{-1})]^\sigma + C_\sigma \int_{(2t)^{-1}}^\infty \frac{\Theta_\sigma(r)}{r} dr \\ &\leq C_\sigma [\log(1/t)]^{1-N} \end{aligned}$$

for $t \in (0, 1/2)$. Finally, since $N > (5/2)$, we may choose $\delta \in (0, 1/4)$ so small that $((2/3) - \delta)(1 - N) < -1$. This leads to

$$\int_0^1 [\eta_\sigma(t)]^{\frac{2}{3}-\delta} \frac{dt}{t} \leq C + C \int_0^{1/2} [\log(1/t)]^{(1-N)(\frac{2}{3}-\delta)} \frac{dt}{t} < \infty,$$

and completes the proof. \square

We are now ready to prove the interior Lipschitz estimates for \mathcal{L}_ε . We first treat the case $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$.

Lemma 4.4. *Suppose that $A(y)$ satisfies the same conditions as in Theorem 1.1. Let $u_\varepsilon \in H^1(2B; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $2B$, where $B = B(x_0, r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Then $|\nabla u_\varepsilon| \in L^\infty(B)$ and*

$$\|\nabla u_\varepsilon\|_{L^\infty(B)} \leq \frac{C}{r} \left\{ \int_{2B} |u_\varepsilon|^2 \right\}^{1/2}, \quad (4.11)$$

where C depends only on A .

Proof. By translation and dilation it suffices to prove that

$$|\nabla u_\varepsilon(0)| \leq C \left\{ \int_{B(0,1)} |u_\varepsilon|^2 \right\}^{1/2}, \quad (4.12)$$

if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(0, 1)$. Note that we only need to treat the case $0 < \varepsilon < (1/4)$, since the case $\varepsilon \geq (1/4)$ follows from the standard local regularity theory for second-order elliptic systems with Hölder continuous coefficients. Let $v_\varepsilon(x) = \varepsilon^{-1}u_\varepsilon(\varepsilon x)$. Then $\mathcal{L}_1(v_\varepsilon) = 0$ in $B(0, 2\varepsilon^{-1})$. By the standard regularity theory for \mathcal{L}_1 ,

$$\begin{aligned} |\nabla u_\varepsilon(0)| &= |\nabla v_\varepsilon(0)| \leq C \inf_{q \in \mathbb{R}^m} \left\{ \int_{B(0,2)} |v_\varepsilon - q|^2 \right\}^{1/2} \\ &= C \inf_{q \in \mathbb{R}^m} \frac{1}{\varepsilon} \left\{ \int_{B(0,2\varepsilon)} |u_\varepsilon - q|^2 \right\}^{1/2}. \end{aligned}$$

To complete the proof we use Theorem 3.2, with $K = 0$, to obtain

$$\frac{1}{\varepsilon} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B(0,2\varepsilon)} |u_\varepsilon - q|^2 \right\}^{1/2} \leq C \left\{ \int_{B(0,1)} |u_\varepsilon|^2 \right\}^{1/2}. \quad (4.13)$$

Note that the condition (3.11) is given by Lemma 4.1, while the condition (3.12) is given by Lemma 4.2. Also, the Dini condition (3.13) is satisfied in view of Lemma 4.3. As a result, the estimate (4.13) follows from (3.14) with $t = 2\varepsilon$. \square

Remark 4.5. In the argument for Lemma 4.4, we used only the first conclusion (3.14) of Theorem 3.2. The second conclusion (3.15) is also useful, and yields the following Liouville result: if $u \in H^1(\mathbb{R}^d; \mathbb{R}^m)$ is any solution of $\mathcal{L}_1(u) = 0$ in \mathbb{R}^d satisfying the linear growth condition

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \left\{ \int_{B(0,r)} |u|^2 \right\}^{1/2} < \infty,$$

then there exists $M \in \mathbb{R}^{m \times d}$ such that

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \left\{ \int_{B(0,r)} |u(x) - Mx|^2 dx \right\}^{1/2} = 0.$$

In other words, if an entire solutions grows at most linearly, it is close to an affine function. To prove this, we follow the argument of Lemma 4.4 with $\varepsilon > 0$ fixed and $u_\varepsilon(x) := \varepsilon u(x/\varepsilon)$. We notice that in the application of Theorem 3.2 we invoked to get (4.13), we also obtain from the second conclusion of the theorem that, for every $\varepsilon < t < 1/4$,

$$\frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B(0,2t)} |u_\varepsilon(x) - Mx - q|^2 dx \right\}^{1/2} \leq C \{t^\alpha + \eta(\varepsilon/t)\} \left\{ \int_{B(0,1)} |u_\varepsilon|^2 \right\}^{1/2}.$$

By undoing the scaling and writing this in terms of u , we obtain, for every $1 < r < 1/4\varepsilon$,

$$\frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B(0,2r)} |u(x) - Mx - q|^2 dx \right\}^{1/2} \leq C \{(\varepsilon r)^\alpha + \eta(1/r)\} \varepsilon \left\{ \int_{B(0,1/\varepsilon)} |u|^2 \right\}^{1/2}.$$

Sending $\varepsilon \rightarrow 0$ and using the growth hypothesis, we get

$$\frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \int_{B(0,2r)} |u(x) - Mx - q|^2 dx \right\}^{1/2} \leq C\eta(1/r).$$

We now obtain the Liouville property by applying the previous inequality on the dyadic scales $r_k := 2^k$, $k \in \mathbb{N}$, and using the Dini condition (3.13) to verify that the sequence $\{M_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$ of corresponding affine approximations is a Cauchy sequence.

As we were finishing the writing of this paper, we became aware of some very recent results of Gloria, Neukamm, and Otto [11], who obtain a more general version of the Liouville result presented above in Remark 4.5. Their scheme is similar to the one from [1], which we use here. Both are based on a Campanato iteration to obtain an improvement of flatness for solutions, although “flatness” in [1], and in this paper, is defined with respect to *affine functions*, while [11], following [2, 4], defines it with respect to *correctors*. The latter notion allows to formulate some more precise results, although it does not seem to help estimating the gradient of the correctors themselves (which is more or less equivalent to the task of obtaining uniform Lipschitz estimates).

Theorem 4.6. *Suppose that $A(y)$ satisfies the same conditions as in Theorem 1.1. Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $2B$, where $B = B(x_0, r)$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(B)} \leq \frac{C}{r} \left\{ \int_{2B} |u_\varepsilon|^2 \right\}^{1/2} + C r^\beta \sup_{\substack{y \in 2B \\ 0 < t < r}} t^{1-\beta} \int_{B(y,t) \cap 2B} |F| \quad (4.14)$$

for any $\beta \in (0, 1)$, where C depends only on β and A .

Proof. By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. We may also assume $d \geq 3$, as the 2-d case may be reduced to the 3-d case by adding a dummy variable.

Consider

$$v_\varepsilon(x) = \int_{2B} \Gamma_\varepsilon(x, y) F(y) dy,$$

where $\Gamma_\varepsilon(x, y)$ denotes the matrix of fundamental solutions for \mathcal{L}_ε in \mathbb{R}^d , with pole at y . Note that $\mathcal{L}_\varepsilon(v_\varepsilon) = F$ in $2B$. By the interior Hölder estimates in [21], we have

$$|\Gamma_\varepsilon(x, y)| \leq C |x - y|^{2-d} \quad \text{for any } x, y \in \mathbb{R}^d, \quad (4.15)$$

where C depends only on A . Since $\mathcal{L}_\varepsilon(\Gamma_\varepsilon(\cdot, y)) = 0$ in $\mathbb{R}^d \setminus \{y\}$, we may use (4.15) and (4.11) to obtain

$$|\nabla_x \Gamma_\varepsilon(x, y)| \leq C |x - y|^{1-d} \quad \text{for any } x, y \in \mathbb{R}^d. \quad (4.16)$$

It is not hard to see that this gives

$$\|\nabla v_\varepsilon\|_{L^\infty(2B)} + \|v_\varepsilon\|_{L^\infty(2B)} \leq C_\beta \sup_{\substack{y \in 2B \\ 0 < t < 1}} t^{1-\beta} \int_{B(y,t) \cap 2B} |F|. \quad (4.17)$$

for any $\beta \in (0, 1)$.

Finally, since $\mathcal{L}_\varepsilon(u_\varepsilon - v_\varepsilon) = 0$ in $2B$, we may invoke Lemma 4.4 to obtain

$$\begin{aligned} \|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B)} &\leq C \left\{ \int_{2B} |u_\varepsilon - v_\varepsilon|^2 \right\}^{1/2} \\ &\leq C \left\{ \int_{2B} |u_\varepsilon|^2 \right\}^{1/2} + C_\beta \sup_{\substack{y \in 2B \\ 0 < t < 1}} t^{1-\beta} \int_{B(y,t) \cap 2B} |F|, \end{aligned}$$

where we have used (4.17) for the last inequality. This, together with (4.17), yields the estimate (4.14). \square

Remark 4.7. Fix $1 \leq j \leq d$, $1 \leq \beta \leq m$, and $y \in \mathbb{R}^d$. Let

$$u(x) = \chi_{T,j}^\beta(x) - \chi_{T,j}^\beta(y) + (x_j - y_j)e^\beta,$$

where $T \geq 1$. Then $\mathcal{L}_1(u) = -T^{-2}\chi_{T,j}^\beta$ in \mathbb{R}^d . It follows from Theorem 4.6 that

$$|\nabla u(y)| \leq \frac{C}{T} \left(\int_{B(y,T)} |u|^2 \right)^{1/2} + C T^{-1} \|\chi_T\|_\infty \leq C.$$

As a result, if A satisfies the same conditions in Theorem 1.1, then

$$\|\nabla \chi_T\|_{L^\infty(\mathbb{R}^d)} \leq C, \quad (4.18)$$

where C depends only on A .

5 Interior $W^{1,p}$ estimates

The goal of this section is to prove the following theorem.

Theorem 5.1. *Suppose that $A(y)$ is uniformly almost-periodic in \mathbb{R}^d and satisfies (1.2). Also assume $A(y)$ satisfies the condition (1.8) for some $N > 5/2$. Let $u_\varepsilon \in H^1(2B; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f)$ in $2B$ for some ball B in \mathbb{R}^d . Suppose that $f = (f_i^\alpha) \in L^p(2B; \mathbb{R}^{dm})$ for some $2 < p < \infty$. Then*

$$\left\{ \int_B |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C_p \left\{ \left(\int_{2B} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\int_{2B} |f|^p \right)^{1/p} \right\}, \quad (5.1)$$

where C_p depends only on p and A .

We remark that in contrast to Theorem 4.6, the Hölder continuity condition (1.7) is not required for $W^{1,p}$ estimates.

We first treat the case where $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$.

Lemma 5.2. *Assume A satisfies the same assumptions as in Theorem 5.1. Let $u_\varepsilon \in H^1(2B; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $2B$, where $B = B(x_0, r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Then $|\nabla u_\varepsilon| \in L^p(B)$ for any $2 < p < \infty$, and*

$$\left\{ \int_B |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C_p \left\{ \int_{2B} |\nabla u_\varepsilon|^2 \right\}^{1/2}, \quad (5.2)$$

where C_p depends only on p and A .

Proof. Fix $2 < p < \infty$. By translation and dilation we may assume $x_0 = 0$ and $r = 1$. By subtracting a constant we may also assume $\int_{2B} u_\varepsilon = 0$. We may further assume that $0 < \varepsilon < (1/4)$, as the case $\varepsilon \geq (1/4)$ follows from the standard local $W^{1,p}$ estimates for second-order elliptic systems with continuous coefficients. By rescaling the same theory also gives

$$\left\{ \int_{B(0,\varepsilon)} |\nabla u_\varepsilon|^p \right\}^{1/p} \leq \frac{C_p}{\varepsilon} \inf_{q \in \mathbb{R}^m} \left\{ \int_{B(0,2\varepsilon)} |u_\varepsilon - q|^2 \right\}^{1/2}. \quad (5.3)$$

An inspection of the proof for Lemma 4.4 shows that estimate (4.13) continues to hold under the assumption in Theorem 5.1 (the Hölder continuity of A is not required). Thus,

$$\left\{ \int_{B(0,\varepsilon)} |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C_p \|u_\varepsilon\|_{L^2(B(0,1))}.$$

By translation this implies that for any $z \in B(0,1)$,

$$\int_{B(z,\varepsilon)} |\nabla u_\varepsilon|^p dx \leq C_p \varepsilon^d \|u_\varepsilon\|_{L^2(B(0,2))}^p.$$

It follows by a simple covering argument that

$$\int_{B(0,1)} |\nabla u_\varepsilon|^p \leq C_p \|u_\varepsilon\|_{L^2(B(0,2))}^p \leq C_p \|\nabla u_\varepsilon\|_{L^2(B(0,2))}^p,$$

where we have used Poincaré inequality for the last step. \square

The reduction of Theorem 5.1 to Lemma 5.2 is done through a refined version of Calderón-Zygmund argument due to Caffarelli and Peral in [6]. Motivated by [6], the following theorem was formulated and proved in [19] (also see [18]).

Theorem 5.3. *Let $F \in L^2(4B_0)$ and $f \in L^p(4B_0)$ for some $2 < p < q < \infty$, where B_0 is a ball in \mathbb{R}^d . Suppose that for each ball $B \subset 2B_0$ with $|B| \leq c_1|B_0|$, there exist two measurable functions F_B and R_B on $2B$, such that $|F| \leq |F_B| + |R_B|$ on $2B$, and*

$$\begin{aligned} \left\{ \int_{2B} |R_B|^q \right\}^{1/q} &\leq C_1 \left\{ \left(\int_{c_2 B} |F|^2 \right)^{1/2} + \sup_{4B_0 \supset B' \supset B} \left(\int_{B'} |f|^2 \right)^{1/2} \right\}, \\ \left\{ \int_{2B} |F_B|^2 \right\}^{1/2} &\leq C_2 \sup_{4B_0 \supset B' \supset B} \left\{ \int_{B'} |f|^2 \right\}^{1/2}, \end{aligned} \quad (5.4)$$

where $C_1, C_2 > 0$, $0 < c_1 < 1$, and $c_2 > 2$. Then $F \in L^p(B_0)$ and

$$\left\{ \int_{B_0} |F|^p \right\}^{1/p} \leq C \left\{ \left(\int_{4B_0} |F|^2 \right)^{1/2} + \left(\int_{4B_0} |f|^p \right)^{1/p} \right\}, \quad (5.5)$$

where C depends only on d, C_1, C_2, c_1, c_2, p , and q .

Proof of Theorem 5.1. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f)$ in $2B_0$ and $f \in L^p(2B_0; \mathbb{R}^{dm})$ for some $2 < p < \infty$. Let $q = p + 1$. We will apply Theorem 5.3 to $F = |\nabla u_\varepsilon|$. For each ball B such that $4B \subset 2B_0$, we write $u_\varepsilon = v_\varepsilon + w_\varepsilon$ on $2B$, where $v_\varepsilon \in H_0^1(4B; \mathbb{R}^{dm})$ is the solution to $\mathcal{L}_\varepsilon(v_\varepsilon) = \operatorname{div}(f)$ in $4B$. Let

$$F_B = |\nabla v_\varepsilon| \quad \text{and} \quad R_B = |\nabla w_\varepsilon|.$$

Then $|F| \leq F_B + R_B$ on $2B$. It is easy to see that the first inequality in (5.4) follows from the energy estimate. Since $\mathcal{L}_\varepsilon(w_\varepsilon) = 0$ in $4B$, it follows from Lemma 5.2 that

$$\begin{aligned} \left\{ \int_{2B} |R_B|^q \right\}^{1/q} &\leq C \left\{ \int_{4B} |R_B|^2 \right\}^{1/2} \\ &\leq C \left\{ \int_{4B} |\nabla u_\varepsilon|^2 \right\}^{1/2} + \left\{ \int_{4B} |\nabla v_\varepsilon|^2 \right\}^{1/2} \\ &\leq C \left\{ \int_{4B} |F|^2 \right\}^{1/2} + C \left\{ \int_{4B} |f|^2 \right\}^{1/2}, \end{aligned}$$

where we have used the energy estimate for the last inequality. This gives the second inequality in (5.4). It then follows by Theorem 5.3 that

$$\left\{ \int_B |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C \left\{ \left(\int_{4B} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\int_{4B} |f|^p \right)^{1/p} \right\}$$

for any ball B such that $4B \subset 2B_0$. By a simple covering argument this gives (5.1) for $B = B_0$. \square

6 Boundary $W^{1,p}$ estimates and proof of Theorems 1.4 and 1.5

In this section we establish uniform boundary $W^{1,p}$ estimate for \mathcal{L}_ε with Dirichlet or Neumann condition. As we shall see, boundary $W^{1,p}$ estimates follow from the interior $W^{1,p}$ estimates and boundary Hölder estimates.

For $r > 0$, let

$$\begin{aligned} D_r &= \{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \phi(x') < x_d < \phi(x') + 10(K_0 + 1)r\}, \\ \Delta_r &= \{(x', \phi(x')) \in \mathbb{R}^d : |x'| < r\}, \end{aligned} \tag{6.1}$$

where $\phi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a $C^{1,\alpha}$ function such that $\text{supp}(\phi) \subset \{x' \in \mathbb{R}^{d-1} : |x'| \leq 10\}$,

$$\phi(0) = 0, \quad \nabla \phi(0) = 0, \quad \text{and} \quad \|\nabla \phi\|_{C^\alpha(\mathbb{R}^{d-1})} \leq K_0 + 1. \tag{6.2}$$

The constant $K_0 > 0$ in (6.2) is fixed. The bounding constants C in the next two lemmas will depend on (α, K_0) , but otherwise not directly on ϕ .

Lemma 6.1. *Suppose that A is uniformly almost-periodic in \mathbb{R}^d and satisfies the ellipticity condition (1.2). Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} , with either $u_\varepsilon = 0$ or $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on Δ_{2r} , for some $0 < r \leq 1$. Then, for any $0 < \beta < 1$,*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_\beta r \left(\frac{|x - y|}{r} \right)^\beta \left(\int_{D_{2r}} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6.3}$$

where C_β depends only on β , A , and (α, K_0) in (6.2).

Proof. In the case of Dirichlet condition $u_\varepsilon = 0$ on Δ_{2r} , the estimate (6.3) was proved in [21] by using a three-step compactness argument introduced in [2]. The compactness argument in [2] for Hölder estimates does not involve correctors and extends readily to the almost-periodic setting. This is also true in the case of Neumann boundary conditions. We omit the details and refer the reader to [14], where uniform boundary Hölder estimates with Neumann conditions were established in the periodic setting. \square

Lemma 6.2. *Suppose that A is uniformly almost-periodic in \mathbb{R}^d and satisfies (1.2). Also assume that the decay condition (1.8) holds for some $C_0 > 0$ and $N > (3/2)$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} , with either $u_\varepsilon = 0$ or $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on Δ_{2r} , for some $0 < r \leq 1$. Then, for any $2 < p < \infty$,*

$$\left\{ \int_{D_r} |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C_p \left\{ \int_{D_{2r}} |\nabla u_\varepsilon|^2 \right\}^{1/2}, \quad (6.4)$$

where C_p depends only on p , A , and (α, K_0) in (6.2).

Proof. By rescaling we may assume $r = 1$. Also assume that $\|\nabla u_\varepsilon\|_{L^2(D_2)} = 1$. Let $\delta(x) = \text{dist}(x, \partial D_2)$. It follows from the interior $W^{1,p}$ estimates in Theorem 5.1 that

$$\begin{aligned} \left(\int_{B(y, \delta(y)/8)} |\nabla u_\varepsilon|^p \right)^{1/p} &\leq C \left(\int_{B(y, \delta(y)/4)} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ &\leq C \left(\int_{B(y, \delta(y)/2)} |u_\varepsilon(x) - u_\varepsilon(y)|^2 dx \right)^{1/2} \\ &\leq C_\beta [\delta(y)]^{\beta-1} \end{aligned} \quad (6.5)$$

for any $\beta \in (0, 1)$, where we have used Lemma 6.1 for the last inequality. By choosing $\beta \in (1 - \frac{1}{p}, 1)$, this implies that

$$\int_{D_1} \left(\int_{B(y, \delta(y)/8)} |\nabla u_\varepsilon(x)|^p dx \right) dy \leq C. \quad (6.6)$$

By Fubini's Theorem we then obtain

$$\int_{D_1} |\nabla u(x)|^p \left\{ \int_{\{y \in D_1 : |y-x| < \frac{\delta(y)}{8}\}} \frac{dy}{[\delta(y)]^d} \right\} dx \leq C. \quad (6.7)$$

Finally, we note that if $|y - x| < \frac{\delta(y)}{8}$, then $\delta(x) \approx \delta(y)$. Also, it is not hard to verify that for $x \in D_1$,

$$D_1 \cap B(x, \delta(x)/16) \subset \{y \in D_1 : |y - x| < \delta(y)/8\}.$$

It follows that

$$\int_{\{y \in D_1 : |y-x| < \frac{\delta(y)}{8}\}} \frac{dy}{[\delta(y)]^d} \geq c > 0.$$

This, together with (6.7), gives

$$\int_{D_1} |\nabla u_\varepsilon(x)|^p dx \leq C,$$

and completes the proof. \square

Theorem 6.3. *Suppose that A is uniformly almost-periodic in \mathbb{R}^d and satisfies (1.2). Also assume that the decay condition (1.8) holds for some $C_0 > 0$ and $N > (3/2)$. Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d for some $\alpha > 0$.*

i) Let $u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h)$ in Ω , where $1 < p < \infty$ and $h \in L^p(\Omega; \mathbb{R}^{m \times d})$. Then

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C_p \|h\|_{L^p(\Omega)}, \quad (6.8)$$

where C_p depends only on p , Ω , and A .

ii) Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h)$ in Ω and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n \cdot h$ on $\partial\Omega$, where $1 < p < \infty$ and $h \in L^p(\Omega; \mathbb{R}^{m \times d})$. Then estimate (6.8) holds with C_p depending only on p , Ω , and A .

Proof. Since the adjoint operator $\mathcal{L}_\varepsilon^*$ satisfies the same conditions as \mathcal{L}_ε , by a duality argument, we may assume that $p > 2$. By a real-variable argument (see [18, 8]), to prove (6.8) for a fixed $p > 2$, it suffices to establish two weak reverse Hölder estimates for some $q > p$:

(i) if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $2B$ and $2B \subset \Omega$, then

$$\left\{ \int_B |\nabla u_\varepsilon|^q \right\}^{1/q} \leq C \left\{ \int_{2B} |\nabla u_\varepsilon|^2 \right\}^{1/2}, \quad (6.9)$$

(ii) if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ on $2B \cap \Omega$ with either $u_\varepsilon = 0$ or $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $2B \cap \partial\Omega$, where $B = B(x_0, r)$, $x_0 \in \partial\Omega$ and $0 < r < r_0 = c_0 \operatorname{diam}(\Omega)$, then

$$\left\{ \int_{B \cap \Omega} |\nabla u_\varepsilon|^q \right\}^{1/q} \leq C \left\{ \int_{2B \cap \Omega} |\nabla u_\varepsilon|^2 \right\}^{1/2}. \quad (6.10)$$

Note that estimate (6.9) is the interior $W^{1,p}$ estimate given by Lemma 5.2, while (6.10) follows from the boundary $W^{1,p}$ estimates proved in Lemma 6.2. \square

We are now in a position to give the proof of Theorems 1.4 and 1.5.

Proof of Theorems 1.4 and 1.5. For the Neumann condition the reduction of Theorem 1.5 to Theorem 6.3 may be found in [8, 14].

For Dirichlet condition the reduction of Theorem 1.4 to Theorem 6.3 is also more or less well known. By considering $u_\varepsilon - w$, where $w \in W^{1,p}(\Omega; \mathbb{R}^m)$ is the solution of $-\Delta w = 0$ in Ω and $w = f$ on $\partial\Omega$, it suffices to prove the theorem for the case $f = 0$. Here we have used the fact that the theorem holds for $\mathcal{L}_\varepsilon = -\Delta$. Next, in view of Theorem 6.3,

we may further assume that $h = 0$. Finally, the case that $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$ may be handled by a duality argument. Indeed, let v_ε be a solution of $\mathcal{L}_\varepsilon^*(v_\varepsilon) = \operatorname{div}(h)$ in Ω and $v_\varepsilon = 0$ on $\partial\Omega$, where $h = (h_i^\alpha) \in C_0^\infty(\Omega; \mathbb{R}^{m \times d})$. Then

$$\begin{aligned} \left| \int_\Omega \frac{\partial u_\varepsilon^\alpha}{\partial x_i} \cdot h_i^\alpha \right| &= \left| \int_\Omega F^\alpha \cdot v^\alpha \right| \leq \|F\|_{L^p(\Omega)} \|v_\varepsilon\|_{L^{p'}(\Omega)} \\ &\leq C \|F\|_{L^p(\Omega)} \|\nabla v_\varepsilon\|_{L^{p'}(\Omega)} \leq C \|F\|_{L^p(\Omega)} \|h\|_{L^{p'}(\Omega)}, \end{aligned}$$

where we have used Poincaré inequality and $W^{1,p}$ estimates for $\mathcal{L}_\varepsilon^*$. By duality this gives $\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \|F\|_{L^p(\Omega)}$. \square

7 Boundary Lipschitz estimates with Dirichlet condition and Proof of Theorems 1.1 and 1.3

In this section we establish the uniform boundary Lipschitz estimates for \mathcal{L}_ε in bounded $C^{1,\alpha}$ domains and give the proof of Theorem 1.1. As in the case of interior Lipschitz estimates, our approach is based on the general scheme outlined in Section 3. However, modifications are needed to take into account the boundary contribution.

Lemma 7.1. *Suppose that $\mathcal{L}_0(w) = 0$ in D_r and $w = f$ on Δ_r for some $0 < r \leq 1$. Let*

$$\begin{aligned} G(t) = & \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^d}} \left\{ \left(\int_{D_t} |w - Mx - q|^2 \right)^{1/2} + \|f - Mx - q\|_{L^\infty(\Delta_t)} \right. \\ & \left. + t \|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_t)} + t^{1+\beta} \|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\beta}(\Delta_t)} \right\} \end{aligned}$$

for $0 < t \leq r$, where $\beta = \alpha/2$. Then, there exists $\theta \in (0, 1/4)$, depending only on μ and (α, K_0) in (6.2), such that

$$G(\theta r) \leq (1/2)G(r). \quad (7.1)$$

Proof. The lemma follows from boundary $C^{1,\alpha}$ estimates for second-order elliptic systems with constant coefficients. By rescaling we may assume $r = 1$. By choosing $q = w(0)$ and $M = \nabla w(0)$, it is easy to see that for any $\theta \in (0, 1/4)$,

$$G(\theta) \leq C \theta^\beta \|w\|_{C^{1,\beta}(D_\theta)}.$$

By boundary $C^{1,\alpha}$ estimates for \mathcal{L}_0 , we obtain

$$\|w\|_{C^{1,\beta}(D_\theta)} \leq C \left\{ \left(\int_{D_1} |w|^2 \right)^{1/2} + \|g\|_{L^\infty(\Delta_1)} + \|\nabla_{\tan} g\|_{L^\infty(\Delta_1)} + \|g\|_{C^{0,\beta}(\Delta_1)} \right\},$$

where C depends only on μ and (α, K_0) . It follows that

$$G(\theta) \leq C \theta^\beta \left\{ \left(\int_{D_1} |w|^2 \right)^{1/2} + \|g\|_{L^\infty(\Delta_1)} + \|\nabla_{\tan} g\|_{L^\infty(\Delta_1)} + \|g\|_{C^{0,\beta}(\Delta_1)} \right\}.$$

Finally, since $\mathcal{L}_0(Mx + q) = 0$ for any $M \in \mathbb{R}^{m \times d}$ and $q \in \mathbb{R}^m$, the estimate above implies that

$$G(\theta) \leq C \theta^\beta G(1).$$

The desired estimate follows by choosing $\theta \in (0, 1/4)$ so small that $C\theta^\beta \leq (1/2)$. \square

Lemma 7.2. *Let $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} and $u_\varepsilon = f$ on Δ_{2r} , where $0 < \varepsilon < r \leq 1$. Then there exists w such that $\mathcal{L}_0(w) = 0$ in D_r , $w = f$ on Δ_r , and*

$$\begin{aligned} \left\{ \int_{D_r} |u_\varepsilon - w|^2 \right\}^{1/2} &\leq C [\omega(\varepsilon/r)]^{\frac{2}{3}-\delta} \left\{ \inf_{q \in \mathbb{R}^m} \left[\left(\int_{D_{2r}} |u_\varepsilon - q|^2 \right)^{1/2} + \|f - q\|_{L^\infty(\Delta_{2r})} \right] \right. \\ &\quad \left. + r \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} + r^{1+\beta} \|\nabla_{\tan} f\|_{C^{0,\beta}(\Delta_{2r})} \right\}, \end{aligned} \quad (7.2)$$

where $\delta \in (0, 1/4)$, $\beta = \alpha/2$, and $\omega(t) = \omega_\sigma(t)$ is defined by (2.20). The constant C depends only on $\delta, \sigma, (\alpha, K_0)$ in (6.2), and A .

Proof. By rescaling we may assume $r = 1$. For each $t \in [0, 1/4]$, we construct a bounded $C^{1,\alpha}$ domain Ω_{1+t} in \mathbb{R}^d such that (1) $D_1 \subset \Omega_1 \subset \Omega_{1+t} \subset D_{3/2}$, (2) there exists a $C^{1,\alpha}$ diffeomorphism $\Lambda_t : \partial\Omega_1 \rightarrow \partial\Omega_{1+t}$ with uniform bounds and the property that $|\Lambda_t(x) - x| \leq Ct$ for any $x \in \partial\Omega_1$, and (3) for each $x \in \partial\Omega_1$, $B(x, ct) \cap D_2 \subset \Omega_{1+t}$.

Let $w = w_t$ be the solution of Dirichlet problem: $\mathcal{L}_0(w) = 0$ in Ω_{1+t} and $w = u_\varepsilon$ on $\partial\Omega_{1+t}$. Note that $\mathcal{L}_0(w) = 0$ in D_1 and $w = f$ on Δ_1 . We will show that w satisfies the estimate (7.2) for some suitable choice of t .

Let v_ε be the solution of $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in Ω_1 and $v_\varepsilon = w$ on $\partial\Omega_1$. Since $\mathcal{L}_\varepsilon(u_\varepsilon - v_\varepsilon) = 0$ in Ω_1 , by the Hölder estimate (1.12) for \mathcal{L}_ε ,

$$\begin{aligned} \|u_\varepsilon - v_\varepsilon\|_{L^2(D_1)} &\leq C \|u_\varepsilon - w\|_{C^\kappa(\partial\Omega_1)} \leq C t^{\gamma-\kappa} \|u_\varepsilon\|_{C^\gamma(D_{3/2})} \\ &\leq C t^{\gamma-\kappa} \left\{ \left(\int_{D_2} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)} \right\}, \end{aligned} \quad (7.3)$$

where $0 < \kappa < \gamma < 1$. The fact that $|\Lambda_t(x) - x| \leq Ct$ for $t \in \partial\Omega_1$ is used for the second inequality in (7.3). Next, by Theorem 2.6, we see that

$$\begin{aligned} \|v_\varepsilon - w\|_{L^2(D_1)} &\leq C [\omega(\varepsilon)]^{2/3} \|w\|_{C^{1,\kappa}(\partial\Omega_1)} \\ &\leq C [\omega(\varepsilon)]^{2/3} t^{\gamma-1-\kappa} \left\{ \|w\|_{C^\gamma(\Omega_{1+t})} + \|g\|_{C^{1,\kappa}(\Delta_2)} \right\} \\ &\leq C [\omega(\varepsilon)]^{2/3} t^{\gamma-1-\kappa} \left\{ \|u_\varepsilon\|_{C^\gamma(\Omega_{1+t})} + \|g\|_{C^{1,\kappa}(\Delta_2)} \right\} \\ &\leq C [\omega(\varepsilon)]^{2/3} t^{\gamma-1-\kappa} \left\{ \left(\int_{D_2} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^{1,\kappa}(\Delta_2)} \right\}, \end{aligned} \quad (7.4)$$

where we have used the boundary $C^{1,\kappa}$ estimates for \mathcal{L}_0 for the second inequality and Hölder estimates for the third. We point out that for the second inequality in (7.4) we

also have used the fact $B(x, ct) \cap D_2 \subset \Omega_{1+t}$ for any $x \in \partial\Omega_1$. It follows from (7.3) and (7.4) that

$$\begin{aligned} \|u_\varepsilon - w\|_{L^2(D_1)} &\leq \|u_\varepsilon - v_\varepsilon\|_{L^2(D_1)} + \|v_\varepsilon - w\|_{L^2(D_1)} \\ &\leq Ct^{\gamma-\kappa} \left\{ 1 + t^{-1} [\omega(\varepsilon)]^{2/3} \right\} \left\{ \left(\int_{D_2} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^{1,\kappa}(\Delta_2)} \right\}, \\ &\leq C [\omega(\varepsilon)]^{\frac{2}{3}-\delta} \left\{ \left(\int_{D_2} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^{1,\kappa}(\Delta_2)} \right\}, \end{aligned}$$

where we have chosen $t = c[\omega(\varepsilon)]^{2/3} \in (0, 1/4)$, $\kappa = (3/4)\delta$ and $\gamma = 1 - (3/4)\delta$. This yields the estimate (7.2), as $\mathcal{L}_\varepsilon(u_\varepsilon - q) = 0$ in D_2 for any $q \in \mathbb{R}^m$. \square

We are now ready to prove the boundary Lipschitz estimates for \mathcal{L}_ε .

Theorem 7.3. *Suppose that $A(y)$ satisfies the same conditions as in Theorem 1.1. Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} and $u_\varepsilon = f$ on Δ_{2r} for some $0 < r \leq 1$. Then*

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(D_r)} &\leq C \left\{ \frac{1}{r} \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r \|f\|_{L^\infty(\Delta_{2r})} \right. \\ &\quad \left. + \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} + r^\beta \|\nabla_{\tan} f\|_{C^{0,\beta}(\Delta_{2r})} \right\}, \end{aligned} \tag{7.5}$$

where $\beta = \alpha/2$ and C depends only on (α, K_0) and A .

Proof. By rescaling we may assume that $r = 1$. Let

$$\begin{aligned} H(t) = & t^{-1} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^d}} \left\{ \left(\int_{D_t} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + \|f - Mx - q\|_{L^\infty(\Delta_t)} \right. \\ & \left. + t \|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_t)} + t^{1+\beta} \|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\beta}(\Delta_t)} \right\}, \end{aligned}$$

where $0 < t \leq 1$. For each $\varepsilon < t < 1$, let $w = w_t$ be a solution of $\mathcal{L}_0(w) = 0$ in D_t with $w = f$ on Δ_t , given by Lemma 7.2. For $0 < s \leq t$, let $G(s)$ be defined as $H(t)$, but with u_ε replaced by w and t replaced by s . Observe that

$$\begin{aligned} H(\theta t) &\leq G(\theta t) + \frac{1}{\theta t} \left\{ \int_{D_{\theta t}} |u_\varepsilon - w|^2 \right\}^{1/2} \\ &\leq \frac{1}{2} G(t) + \frac{1}{\theta t} \left\{ \int_{D_{\theta t}} |u_\varepsilon - w|^2 \right\}^{1/2} \\ &\leq \frac{1}{2} H(t) + \frac{C}{t} \left\{ \int_{D_t} |u_\varepsilon - w|^2 \right\}^{1/2}, \end{aligned}$$

where we have used Lemma 7.1 for the second inequality. This, together with Lemma 7.2, gives

$$H(\theta t) \leq \frac{1}{2}H(t) + C[\omega(\varepsilon/t)]^{\frac{2}{3}-\delta} \left\{ t^{-1} \inf_{q \in \mathbb{R}^m} \left[\left(\int_{D_{2t}} |u_\varepsilon - q|^2 \right)^{1/2} + \|f - q\|_{L^\infty(\Delta_{2t})} \right] \right. \\ \left. + \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2t})} + t^\beta \|\nabla_{\tan} f\|_{C^{0,\beta}(\Delta_{2t})} \right\} \quad (7.6)$$

for any $\varepsilon < t \leq 1$. Let $r_j = \theta^{j+1}$ for $0 \leq j \leq \ell$, where ℓ is chosen so that $\theta^{\ell+2} < \varepsilon \leq \theta^{\ell+1}$. Let

$$F_j = H(r_j) \quad \text{and} \quad p_j = |M_j|,$$

where $M_j \in \mathbb{R}^{m \times d}$ is a matrix such that

$$H(r_j) = r_j^{-1} \inf_{q \in \mathbb{R}^m} \left\{ \left(\int_{D_{r_j}} |u_\varepsilon - M_j x - q|^2 \right)^{1/2} + \|f - M_j x - q\|_{L^\infty(\Delta_{r_j})} \right. \\ \left. + r_j \|\nabla_{\tan}(f - M_j x - q)\|_{L^\infty(\Delta_{r_j})} + r_j^{1+\beta} \|\nabla_{\tan}(f - M_j x - q)\|_{C^{0,\beta}(\Delta_{r_j})} \right\}.$$

In view of (7.6) we obtain

$$F_{j+1} \leq \frac{1}{2}F_j + C[\omega(\varepsilon\theta^{-j-1})]^{\frac{2}{3}-\delta} \{F_{j-1} + p_{j-1}\}. \quad (7.7)$$

Also observe that since D_r satisfies the interior cone condition,

$$|M_{j+1} - M_j| \leq \frac{C}{r_{j+1}} \inf_{q \in \mathbb{R}^m} \left\{ \int_{D_{r_{j+1}}} |(M_{j+1} - M_j)x - q|^2 \right\}^{1/2} \\ \leq \frac{C}{r_{j+1}} \inf_{q \in \mathbb{R}^m} \left\{ \int_{D_{r_{j+1}}} |u_\varepsilon - M_{j+1}x - q|^2 \right\}^{1/2} + \frac{C}{r_{j+1}} \inf_{q \in \mathbb{R}^m} \left\{ \int_{D_{r_{j+1}}} |u_\varepsilon - M_jx - q|^2 \right\}^{1/2} \\ \leq C(F_j + F_{j+1}).$$

It follows that

$$p_{j+1} = |M_{j+1}| \leq |M_j| + C(F_j + F_{j+1}) = p_j + C(F_j + F_{j+1}). \quad (7.8)$$

Recall that the condition (1.8) implies that

$$\sum_{j=0}^{\ell} [\omega(\varepsilon\theta^{-j-1})]^{\frac{2}{3}-\delta} \leq C \int_0^1 [\omega(t)]^{\frac{2}{3}-\delta} \frac{dt}{t} < \infty,$$

for some $\sigma, \delta \in (0, 1)$. This allows us to apply Lemma 3.1 to obtain

$$F_j + p_j \leq C \{p_0 + F_0 + F_1\} \\ \leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\beta}(\Delta_1)} \right\}$$

for any $0 \leq j \leq \ell$. As a result, we see that for any $\varepsilon < t < 1/4$,

$$\begin{aligned} \left(\int_{D_t} |\nabla u_\varepsilon|^2 \right)^{1/2} &\leq \frac{C}{t} \inf_{q \in \mathbb{R}^m} \left\{ \left(\int_{D_{2t}} |u_\varepsilon - q|^2 \right)^{1/2} + \|f - q\|_{L^\infty(\Delta_{2t})} \right\} + C \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2t})} \\ &\leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\beta}(\Delta_1)} \right\}, \end{aligned} \quad (7.9)$$

where we have used Caccioppoli's inequality for the first inequality.

Finally, since $A(y)$ is Hölder continuous, we may use apply the classical boundary Lipschitz estimates for \mathcal{L}_1 and a blow-up argument to obtain

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(D_\varepsilon)} &\leq \frac{C}{\varepsilon} \inf_{q \in \mathbb{R}^m} \left\{ \left(\int_{D_{2\varepsilon}} |u_\varepsilon - q|^2 \right)^{1/2} + \|f - q\|_{L^\infty(\Delta_{2\varepsilon})} \right\} + C \|\nabla_{\tan} f\|_{C^\sigma(\Delta_{2\varepsilon})} \\ &\leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\beta}(\Delta_1)} \right\}, \end{aligned}$$

where we have used (7.9) with $t = 2\varepsilon$ for the second inequality. Consequently, we see that

$$\left(\int_{D_t} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\beta}(\Delta_1)} \right\} \quad (7.10)$$

holds for any $0 < t < 1/4$. This, together with the interior Lipschitz estimates proved in Section 4, yields

$$\|\nabla u_\varepsilon\|_{L^\infty(D_1)} \leq C \left\{ \left(\int_{D_2} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\beta}(\Delta_2)} \right\}. \quad (7.11)$$

The proof is complete. \square

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to show that if $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_{2r} and $u_\varepsilon = f$ on Δ_{2r} for some $0 < r < 1$, then

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(D_r)} &\leq Cr^{-1} \|u_\varepsilon\|_{L^\infty(D_{2r})} + C \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} + Cr^\beta \|\nabla_{\tan} f\|_{C^{0,\beta}(\Delta_{2r})} \\ &\quad + Cr^\beta \sup_{\substack{x \in D_{2r} \\ 0 < t < r}} t^{1-\beta} \int_{B(x,t) \cap D_{2r}} |F|. \end{aligned} \quad (7.12)$$

Estimate (1.9) follows from (7.12) and the interior Lipschitz estimate by a simple covering argument.

To prove (7.12), we may assume that $r = 1$ and $d \geq 3$. The case $F = 0$ is already proved in the last lemma. The general case may be handled by the use of Green functions.

Indeed, let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d such that $D_{3/2} \subset \Omega \subset D_2$. Let $G_\varepsilon(x, y)$ denote the matrix of Green functions for \mathcal{L}_ε in Ω , with pole at y . By the boundary Hölder estimates in [21], we know $|G_\varepsilon(x, y)| \leq C |x - y|^{2-d}$ for any $x, y \in \Omega$. Since $\mathcal{L}_\varepsilon(G_\varepsilon(\cdot, y)) = 0$ in $\Omega \setminus \{y\}$ and $G(x, y) = 0$ for $x \in \partial\Omega$, we may use the boundary Lipschitz estimate in the last lemma to show that $|\nabla_x G_\varepsilon(x, y)| \leq C |x - y|^{1-d}$ for any $x, y \in \Omega$. One then considers $u_\varepsilon - v_\varepsilon$ in D_2 , where $v_\varepsilon(x) = \int_\Omega G_\varepsilon(x, y) F(y) dy$. The rest of the argument is similar to that in the proof of Theorem 4.6. We omit the details. \square

Proof of Theorem 1.3. The non-tangential maximal function of u_ε is defined by

$$(u_\varepsilon)^*(Q) = \sup \{ |u_\varepsilon(x)| : x \in \Omega \text{ and } |x - Q| < C_0 \text{dist}(x, \partial\Omega) \} \quad (7.13)$$

for $Q \in \partial\Omega$, where $C_0 = C_0(\Omega)$ is sufficiently large. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω and $u_\varepsilon = f$ on $\partial\Omega$. It is well known that the estimate (1.18) implies that $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\partial\Omega)}$, and

$$(u_\varepsilon)^*(Q) \leq C \mathcal{M}_{\partial\Omega}(f)(Q) \quad \text{for any } Q \in \partial\Omega,$$

where $\mathcal{M}_{\partial\Omega}(f)$ denotes the Hardy-Littlewood maximal function of f on $\partial\Omega$. It follows that

$$\|(u_\varepsilon)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}$$

for any $1 < p < \infty$. \square

8 Boundary Lipschitz estimates with Neumann conditions and proof of Theorem 1.2

In this section we establish the uniform Lipschitz estimates with Neumann boundary conditions and give the proof of Theorem 1.2. Throughout the section we will assume that A satisfies the same conditions as in Theorem 1.2.

Let D_r and Δ_r be defined as in (6.1) and (α, K_0) given in (6.2).

Lemma 8.1. *Suppose that $\mathcal{L}_0(w) = 0$ in D_{2r} and $\frac{\partial w}{\partial \nu_0} = g$ on Δ_{2r} . Let*

$$\begin{aligned} \Psi(t) = & \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^d}} \left\{ \left(\int_{D_t} |w - Mx - q|^2 \right)^{1/2} + t \left\| \frac{\partial}{\partial \nu_0} (w - Mx) \right\|_{L^\infty(\Delta_t)} \right. \\ & \left. + t^{1+\beta} \left\| \frac{\partial}{\partial \nu_0} (w - Mx) \right\|_{C^{0,\beta}(\Delta_t)} \right\}, \end{aligned}$$

for $0 < t \leq r$, where $\beta = \alpha/2$. Then there exists $\theta \in (0, 1/4)$, depending only on μ, α and K_0 , such that

$$\Psi(\theta r) \leq (1/2)\Psi(r). \quad (8.1)$$

Proof. The lemma follows from boundary $C^{1,\alpha}$ estimates with Neumann conditions for second-order elliptic systems with constant coefficients. The argument is similar to that in the case of Dirichlet condition. We leave the details to the reader. \square

Lemma 8.2. *Let $\sigma \in (0, 1)$ and*

$$\eta(t) = \eta_\sigma(t) = \left\{ \Theta_\sigma(t^{-1}) + \sup_{T \geq t^{-1}} \langle |\psi - \nabla \chi_T| \rangle \right\}^{1/2}. \quad (8.2)$$

Suppose that there exist $C_0 > 0$ and $N > 3$ such that $\rho(R) \leq C_0 [\log R]^{-N}$ for all $R \geq 2$. Then $\int_0^1 \eta(t) \frac{dt}{t} < \infty$.

Proof. Recall that if there exist $C_0 > 0$ and $N > 1$ such that $\rho(R) \leq C_0 [\log R]^{-N}$ for all $R \geq 2$, then

$$\Theta_\sigma(T) \leq C_\sigma [\log T]^{-N} \quad \text{and} \quad \langle |\psi - \nabla \chi_T| \rangle \leq C [\log T]^{-N+1}$$

for all $T \geq 2$ (see the proof of Lemma 4.3). This gives

$$\eta(t) \leq C [\log(1/t)]^{(1-N)/2} \quad \text{for } t \in (0, 1/2),$$

from which the lemma follows readily. \square

Lemma 8.3. *Suppose that A satisfies the same conditions as in Theorem 1.1. Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on Δ_{2r} , where $0 < \varepsilon < r \leq 1$ and $g \in C^{1,\beta}(\Delta_{2r})$. Then there exists $w \in H^1(D_r; \mathbb{R}^m)$ such that $\mathcal{L}_0(w) = 0$ in D_r , $\frac{\partial w}{\partial \nu_0} = g$ on Δ_{2r} , and*

$$\left\{ \int_{D_r} |u_\varepsilon - w|^2 \right\}^{1/2} \leq C \eta\left(\frac{\varepsilon}{r}\right) \left\{ \inf_{q \in \mathbb{R}^m} \left(\int_{D_{2r}} |u_\varepsilon - q|^2 \right)^{1/2} + r \|g\|_{L^\infty(\Delta_{2r})} \right\} \quad (8.3)$$

where $\beta = \alpha/2$ and $\eta(t)$ is given by (8.2). The constant C depends only on α , K_0 , and A .

Proof. By rescaling we may assume $r = 1$. By subtracting a constant we may assume that $\int_{D_2} u_\varepsilon = 0$. Using Cacciopoli's inequality

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \leq C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{\Delta_2} |g|^2 \right\} \quad (8.4)$$

and the co-area formula, it is not hard to see that there exists a $C^{1,\alpha}$ domain Ω such that $D_1 \subset \Omega \subset D_{3/2}$ and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \leq C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{\Delta_2} |g|^2 \right\}. \quad (8.5)$$

Now let w be the weak solution of $\mathcal{L}_\varepsilon(w) = 0$ in Ω with $\frac{\partial w}{\partial \nu_0} = \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}$ on $\partial\Omega$ and $\int_\Omega w = \int_\Omega u_\varepsilon$. It follows from Theorem 2.9

$$\|u_\varepsilon - w\|_{L^2(\Omega)} \leq C \eta(\varepsilon) \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \leq C \eta(\varepsilon) \left\{ \|u_\varepsilon\|_{L^2(D_2)} + \|g\|_{L^2(\Delta_2)} \right\}, \quad (8.6)$$

where we have used (8.5) for the last inequality. Since $\int_{D_2} u_\varepsilon = 0$, this yields (8.3). \square

Theorem 8.4. Suppose that A satisfies the same conditions as in Theorem 1.2. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on Δ_{2r} . Then

$$\|\nabla u_\varepsilon\|_{L^\infty(D_r)} \leq C \left\{ \left(\int_{D_{2r}} |\nabla u_\varepsilon|^2 \right)^{1/2} + \|g\|_{L^\infty(\Delta_{2r})} + r^\beta \|g\|_{C^{0,\beta}(\Delta_{2r})} \right\}, \quad (8.7)$$

where $\beta = \alpha/2$ and C depends only on (α, K_0) and A .

Proof. With Lemmas 8.1, 8.2 and 8.3 at our disposal, the theorem follows by the same line of argument as in the case of Dirichlet condition. By rescaling we may assume $r = 1$. Let

$$\begin{aligned} \Phi(t) = & t^{-1} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left\{ \left(\int_{D_t} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + t \left\| g - \frac{\partial}{\partial \nu_0}(Mx) \right\|_{L^\infty(\Delta_t)} \right. \\ & \left. + t^{1+\beta} \left\| g - \frac{\partial}{\partial \nu_0}(Mx) \right\|_{C^{0,\beta}(\Delta_t)} \right\} \end{aligned}$$

for $0 < t \leq 1$. For each $\varepsilon < t \leq 1$, let $w = w_t$ be the solution of $\mathcal{L}_0(w) = 0$ in D_t with $\frac{\partial w}{\partial \nu_0} = g$ on Δ_t , given by Lemma 8.3. As in the case of Dirichlet condition, it follows from Lemma 8.1 that

$$\Phi(\theta t) \leq \frac{1}{2} \Phi(t) + \frac{C}{t} \left\{ \int_{D_t} |u_\varepsilon - w|^2 \right\}^{1/2},$$

where $\theta \in (0, 1/4)$ is given by Lemma 8.1. In view of Lemma 8.3, this leads to

$$\Phi(\theta t) \leq \frac{1}{2} \Phi(t) + C \eta(\varepsilon/t) \left\{ \frac{1}{t} \inf_{q \in \mathbb{R}^m} \left(\int_{D_{2t}} |u_\varepsilon - q|^2 \right)^{1/2} + \|g\|_{L^\infty(\Delta_{2t})} \right\}. \quad (8.8)$$

Now, let $r_j = \theta^{j+1}$ for $0 \leq j \leq \ell$, where ℓ is chosen so that $\theta^{\ell+1} < \varepsilon \leq \theta^\ell$. Let

$$F_j = \Phi(r_j) \quad \text{and} \quad p_j = |M_j|,$$

where $M_j \in \mathbb{R}^{m \times d}$ is a matrix such that

$$\begin{aligned} \Phi(r_j) = & r_j^{-1} \left\{ \inf_{q \in \mathbb{R}^m} \left(\int_{D_{r_j}} |u_\varepsilon - M_j x - q|^2 \right)^{1/2} + r_j \left\| g - \frac{\partial}{\partial \nu_0}(M_j x) \right\|_{L^\infty(\Delta_{r_j})} \right. \\ & \left. + r_j^{1+\beta} \left\| g - \frac{\partial}{\partial \nu_0}(M_j x) \right\|_{C^{0,\beta}(\Delta_{r_j})} \right\}. \end{aligned}$$

It follows from the estimate (8.8) that

$$F_{j+1} \leq \frac{1}{2} F_j + C \eta(\varepsilon 2^{-j-1}) \{F_{j-1} + p_{j-1}\}. \quad (8.9)$$

As in the proof of Theorem 7.3, we also have

$$p_{j+1} \leq p_j + C \{F_j + F_{j+1}\}. \quad (8.10)$$

Furthermore, by Lemma 8.2,

$$\sum_{j=1}^{\ell} \eta(\varepsilon \theta^{-j-1}) \leq C \int_0^1 \eta(t) \frac{dt}{t} < \infty.$$

Consequently, we may apply Lemma 3.1 to obtain

$$\begin{aligned} F_j + p_j &\leq C \{p_0 + F_0 + F_1\} \\ &\leq C \left\{ \left(\oint_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^\beta(\Delta_1)} \right\}. \end{aligned}$$

This, together with the Cacciopoli's inequality, yields that for any $\varepsilon < t < (1/4)$,

$$\left\{ \oint_{D_t} |\nabla u_\varepsilon|^2 \right\}^{1/2} \leq C \left\{ \left(\oint_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^\beta(\Delta_1)} \right\}. \quad (8.11)$$

As in the case of Dirichlet condition, we may use a blow-up argument and (8.11) to show that the estimate above in fact holds for any $0 < t < (1/4)$. Finally, we observe that the estimate (8.7) follows from (8.11) and the interior Lipschitz estimates. \square

Remark 8.5. Let Ω be a bounded $C^{1,\alpha}$ domain in \mathbb{R}^d . Let $N_\varepsilon(x, y)$ denote the matrix of Neumann functions for \mathcal{L}_ε in Ω , with pole at y ; i.e.,

$$\begin{cases} \mathcal{L}_\varepsilon \{N_\varepsilon(\cdot, y)\} = I_{m \times m} \delta_y(x) & \text{in } \Omega, \\ \frac{\partial}{\partial \nu_\varepsilon} \{N_\varepsilon(\cdot, y)\} = -|\partial\Omega|^{-1} I_{m \times m} & \text{on } \partial\Omega, \end{cases}$$

where $I_{m \times m}$ denotes the $m \times m$ identity matrix. Suppose that A satisfies the conditions in Theorem 1.2. Since A^* also satisfies the same conditions, it follows from Theorem 8.4 that if $d \geq 3$,

$$\begin{cases} |N_\varepsilon(x, y)| \leq C |x - y|^{2-d}, \\ |\nabla_x N_\varepsilon(x, y)| + |\nabla_y N_\varepsilon(x, y)| \leq C |x - y|^{1-d}, \\ |\nabla_x \nabla_y N_\varepsilon(x, y)| \leq C |x - y|^{-d} \end{cases} \quad (8.12)$$

for any $x, y \in \Omega$, $x \neq y$, where C depends only on A and Ω . We refer the reader to [14] for the proof in the periodic setting.

We now give the proof of Theorem 1.2

Proof of Theorem 1.2. It suffices to show that if $\mathcal{L}(u_\varepsilon) = F$ in D_{2r} and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on $\partial\Omega$ for some $0 < r < 1$, then

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(D_r)} &\leq C \left(\oint_{D_{2r}} |\nabla u_\varepsilon|^2 \right)^{1/2} + C \|g\|_{L^\infty(\Delta_{2r})} + C r^\beta \|g\|_{C^{0,\beta}(\Delta_{2r})} \\ &\quad + C r^\beta \sup_{\substack{x \in D_{2r} \\ 0 < t < r}} t^{1-\beta} \oint_{B(x,t) \cap D_{2r}} |F|. \end{aligned} \quad (8.13)$$

By rescaling we may assume $r = 1$. The case $F = 0$ is given by Theorem 8.4. To deal with the general case, we assume $d \geq 3$ (the case $d = 2$ is reduced to the case $d = 3$ by adding a dummy variable). Let Ω be a bounded $C^{1,\alpha}$ domain such that $D_{3/2} \subset \Omega \subset D_2$. Let $N_\varepsilon(x, y)$ denote the matrix of Neumann functions for \mathcal{L} in Ω , with pole at y . Let $v_\varepsilon(x) = \int_\Omega N_\varepsilon(x, y) F(y) dy$. Note that by (8.12),

$$|\nabla v_\varepsilon(x)| \leq C \int_\Omega \frac{|F(y)|}{|x - y|^{d-1}} dy \leq C \sup_{\substack{x \in D_2 \\ 0 < t < 1}} t^{1-\sigma} \int_{B(x,t) \cap D_2} |F|.$$

By considering $u_\varepsilon - v_\varepsilon$, we may reduce the general case to the case $F = 0$. \square

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Scott N. Armstrong
 Ceremade (UMR CNRS 7534)
 Université Paris-Dauphine
 Paris, France

Zhongwei Shen
 Department of Mathematics
 University of Kentucky
 Lexington, Kentucky 40506, USA.
 E-mail: zshen2@uky.edu

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